MEASUREMENT SCIENCE REVIEW, Volume 2, Section 1, 2002

PROPER ROUNDING OF THE MEASUREMENT RESULTS UNDER THE ASSUMPTION OF UNIFORM DISTRIBUTION[1]

Gejza Wimmer*, Viktor Witkovský**

*Faculty of Natural Sciences, Matej Bel University
Tajovského 40, 974 01 Banská Bystrica, Slovak Republic, and
Mathematical Institute, Slovak Academy of Sciences
Štefánikova 49, 814 73 Bratislava, Slovak Republic
E-mail: wimmer@mat.savba.sk

**Institute of Measurement Science, Slovak Academy of Sciences
Dúbravská cesta 9, 842 19 Bratislava, Slovak Republic
E-mail: umerwitk@savba.sk

Abstract

In the paper we describe a new application of the ε-proper rounding method to measured values and their uncertainties and describe exactly the probability properties of the rounded measured values and their "ε-proper confidence intervals". We propose general rules for proper rounding of the measurement results under uniform distribution assumptions.

Keywords: Rounding errors; Rounding rules; Properly rounded result; Uniform distribution.
AMS classification: 62F25 62F99

1. Introduction

Let us consider the result of the measurement given in the form

$$x \pm s, \tag{1}$$

where x is a realization of a continuous random variable X with mean μ (the unknown measured quantity) and dispersion σ^2, s^2 is the estimate of dispersion σ^2. If σ, the standard deviation is known, instead of (1) we will use the notation $x \pm \sigma$. According to [1], σ is also called standard uncertainty.

In this paper we study from a probabilistic point of view the effect of rounding the measured values and their uncertainties. Throughout the paper we will assume that the measurement errors follow a uniform

[1]The paper was supported by grant from Scientific Grant Agency of the Slovak Republic VEGA 1/7295/20.

distribution. Our aim is to give mathematically correct answer to the following question: How to suitably round and report the result of measurement, given in the form (1), in such a way that the approximation error caused by the rounding procedure leads at most to small, well defined and controlled deviation, say ε, from the nominal probability (significance level) if the standard statistical inference is applied. The solution to this problem is of special interest for the producers of the measurement devices as well as for the experimenters who report the results of measurement which are to be implemented in further measuring process or analysis. The concept called "ε-properly rounded result" was first introduced in [3] where the probability properties of the rounded measured values and their ε-proper confidence intervals were derived under normality assumptions on the distribution of the measurements. The properties of the rounded measured values under triangular distribution are studied in [4].

2. ε-properly rounded result

Consider a continuous random variable X with mean μ and dispersion σ^2. So

$$\xi = \frac{X-\mu}{\sigma}$$

is the standardized continuous random variable with zero mean and unit dispersion.

If the continuous random variable X is uniformly distributed within an arbitrary finite interval (a, b), i.e. $X \sim Un(a, b)$, then the random variable ξ has uniform distribution with an expectation value $E(\xi) = 0$ and variance $\mathrm{Var}(\xi) = 1$, i.e. $\xi \sim Un(-\sqrt{3}, \sqrt{3})$.

Further, for $\alpha \in (0, 1)$ we have

$$P\left\{a_{\frac{\alpha}{2}} < \xi < a_{1-\frac{\alpha}{2}}\right\} = 1-\alpha \tag{2}$$

where $a_{\frac{\alpha}{2}}$, $a_{1-\frac{\alpha}{2}}$ are $\frac{\alpha}{2}$ and $1-\frac{\alpha}{2}$ quantiles, respectively, of the distribution of ξ. So,

$$P\left\{a_{\frac{\alpha}{2}} < \frac{X-\mu}{\sigma} < a_{1-\frac{\alpha}{2}}\right\} = 1-\alpha,$$

or

$$P\left\{X - \sigma\, a_{1-\frac{\alpha}{2}} < \mu < X - \sigma\, a_{\frac{\alpha}{2}}\right\} = 1-\alpha.$$

The $(1-\alpha)$-confidence interval for μ is then defined as a random interval

$$\left\langle X - \sigma\, a_{1-\frac{\alpha}{2}}, X - \sigma\, a_{\frac{\alpha}{2}}\right\rangle. \tag{3}$$

If x is a realization of the continuous random variable X then the interval

$$\left\langle x - \sigma\, a_{1-\frac{\alpha}{2}}, x - \sigma\, a_{\frac{\alpha}{2}}\right\rangle \tag{4}$$

is the interval estimate for μ and is a realization of the $(1-\alpha)$-confidence interval (3). Note that the interval (3) is a random interval (the interval *estimator*) and that the interval (4) is a nonrandom interval (the interval *estimate*). For more details see e.g. [2].

The cummulative distribution function (cdf) of ξ is given by

$$F_\xi(x) = \begin{cases} 0 & \text{if } x \leq -\sqrt{3}, \\ \frac{x+\sqrt{3}}{2\sqrt{3}} & \text{if } -\sqrt{3} < x < \sqrt{3}, \\ 1 & \text{if } x \geq \sqrt{3}. \end{cases} \tag{5}$$

On the other hand, for $\alpha \in (0,1)$ the inverse of $F_\xi(a)$ (the quantile function) is given by

$$a_\alpha = F_\xi^{-1}(\alpha) = \sqrt{3}(2\alpha - 1). \tag{6}$$

First, we introduce two different types of rounding. In the following we will assume standard decimal notation.

Definition 1. *We say that w_* is **rounded to $\mathbf{n}$ significant digits** (or $\mathbf{n}$ **significant digit rounding**) of the value w, where $n \in \{1,2,\ldots\}$, if the following rules apply:*

1. *If the $(n+1)$-st digit in the decimal notation of w is $0,1,2,3$ or 4 then the first n digits in the notation of w_* remain unchanged, and the remaining digits are zero.*
2. *If the $(n+1)$-st digit in the decimal notation of w is $5,6,7,8$ or 9 then the n-th digit in the notation of w_* is increased by 1 and the remaining digits are zero.*

Corollary 1. *If w is rounded to 1 significant digit then*

$$\frac{2}{3} < \frac{w_*}{w} \leq \frac{4}{3}. \tag{7}$$

If w is rounded to 2 significant digits then

$$\frac{20}{21} < \frac{w_*}{w} \leq \frac{22}{21}. \tag{8}$$

If w is rounded to 3 significant digits

$$\frac{200}{201} < \frac{w_*}{w} \leq \frac{202}{201}. \tag{9}$$

Definition 2. *Let $\ldots d_2 10^2 + d_1 10^1 + d_0 10^0 + d_{-1} 10^{-1} + d_{-2} 10^{-2} \ldots$ be the decimal expansion of $|w|$. We say that w_* is **rounded to order $\mathbf{m}$** of the value w, where $m \in \{\ldots,-2,-1,0,1,2,\ldots\}$, if the following rounding rules apply:*

1. *If the digit d_m in the decimal notation of $|w|$ is 0, 1, 2, 3 or 4 then all digits d_n of $|w_*|$ with $n \leq m$ are equal to zero and the remaining digits are unchanged.*

2. *If the digit d_m in the decimal notation of $|w|$ is 5, 6, 7, 8, or 9 then all digits d_n of $|w_*|$ with $n \leq m$ are equal to zero, the d_{m+1} digit is increased by one (with necessary carry over).*

Corollary 2. *If w is rounded to order m then*

$$|w_* - w| < 5 \times 10^m. \tag{10}$$

Assume that the standard uncertainty σ is known and let σ_* be the approximate value of σ rounded to n significant digits (according to Definition 1), then we denote

$$\gamma = \frac{\sigma_*}{\sigma}.$$

Let x be a realization of continuous random variable X and x_* be the rounded value to order m of x (according to Definition 2). Let us introduce X_* the random variable whose realization is x_*. We shall say that X_* is the *rounded value to order* m of X.

The approximate standardized continuous random variable is defined as $\xi_* = (X_* - \mu)/\sigma_*$. However, the probability $P\{a_{\frac{\alpha}{2}} < \xi_* < a_{1-\frac{\alpha}{2}}\}$ is no more equal to $1 - \alpha$ as is given in (2). The true probability will be denoted as $1 - \alpha_*$ and the following relations hold true:

$$\begin{aligned} 1 - \alpha_* &= P\left\{a_{\frac{\alpha}{2}} < \frac{X_* - \mu}{\sigma_*} < a_{1-\frac{\alpha}{2}}\right\} \\ &= P\left\{\gamma a_{\frac{\alpha}{2}} + \frac{X - X_*}{\sigma_*}\gamma < \xi < \gamma a_{1-\frac{\alpha}{2}} + \frac{X - X_*}{\sigma_*}\gamma\right\} \\ &= P\left\{\gamma a_{\frac{\alpha}{2}} + \Delta < \xi < \gamma a_{1-\frac{\alpha}{2}} + \Delta\right\}, \end{aligned} \tag{11}$$

where ξ is the standardized continuous random variable and $\Delta = \frac{X - X_*}{\sigma} = \frac{X - X_*}{\sigma_*} \times \frac{\sigma_*}{\sigma}$.

Definition 3. *Let fixed (small) $\varepsilon > 0$ be the allowed maximum deviation from the nominal significance level $1 - \alpha$ due to rounding. We say that the rounded result of the measurement experiment $x_* \pm \sigma_*$ is ε-properly rounded if $1 - \alpha_* \geq 1 - \alpha - \varepsilon$ for all $\alpha \in (0, 1)$ and α_* given in (11). The ε-properly rounded result will be denoted by ${}_\varepsilon x \pm {}_\varepsilon\sigma$.*

It follows from (11) that for arbitrary $\alpha \in (0, 1)$ the random interval

$$\langle {}_\varepsilon X - {}_\varepsilon\sigma a_{1-\frac{\alpha}{2}}, {}_\varepsilon X - {}_\varepsilon\sigma a_{\frac{\alpha}{2}} \rangle \tag{12}$$

(where ${}_\varepsilon X$ is the continuous random variable whose realization is ${}_\varepsilon x$) contains the true value of μ with probability $1 - \alpha_*$ where $1 - \alpha_* \geq 1 - \alpha - \varepsilon$. We shall call it ε-*proper* $(1 - \alpha)$-*confidence interval* for the (true) value μ.

From the inequality $1-\alpha_* \geq 1-\alpha-\varepsilon$ it follows that the random interval (12) contains the (true) value μ with probability *at least* $1-\alpha-\varepsilon$.

The interval

$$\langle {}_\varepsilon x - {}_\varepsilon\sigma a_{1-\frac{\alpha}{2}},\, {}_\varepsilon x - {}_\varepsilon\sigma a_{\frac{\alpha}{2}} \rangle \tag{13}$$

is a realization of the ε-proper $(1-\alpha)$-confidence interval (12).

For given $\alpha \in (0,1)$, $\gamma \in (\frac{2}{3}, \frac{4}{3})$ (see Corollary 1) and $\delta \geq 0$ the function

$$\lambda(\gamma,\alpha,\delta) = P\left\{\gamma a_{\frac{\alpha}{2}} + \delta < \xi < \gamma a_{1-\frac{\alpha}{2}} + \delta\right\} \tag{14}$$

is a non-increasing continuous function of $\delta \geq 0$ (where ξ is the standardized continuous random variable) attaining its maximum at $\delta = 0$ (note, that the maximum could be attained at many different points) with

$$\lambda(\gamma,\alpha,0) = F_\xi(\gamma a_{1-\frac{\alpha}{2}}) - F_\xi(\gamma a_{\frac{\alpha}{2}}), \tag{15}$$

where F_ξ denotes the cdf of the standardized uniform distribution. Furthermore,

$$\lim_{\delta\to\infty} \lambda(\gamma,\alpha,\delta) = 0.$$

In order to derive the ε-properly rounded result in this situation we compute $\lambda(\gamma,\alpha,0)$. According to (15) for arbitrary $\gamma \in (\frac{2}{3}; 1\rangle$ we obtain

$$\sup_{\alpha\in(0,1)} \{(1-\alpha) - (1-\alpha^*)\} \geq \sup_{\alpha\in(0,1)} \{(1-\alpha) - \lambda(\gamma,\alpha,0)\} = \underline{\lambda}(\gamma) = 1-\gamma, \tag{16}$$

(see Appendix, $\sup$ means the supremum operator).

If $\underline{\lambda}(\gamma) > \varepsilon$ for chosen (small) $\varepsilon > 0$ and given $\gamma \in (\frac{2}{3}; 1)$ then, according to (15) and Definition 3, there does not exist the ε-properly rounded result even in the case $X_* = X$ (i.e. if we always use the exact, not a rounded result x).

The threshold value γ^*_ε of γ, for the given $\varepsilon > 0$, is given as a solution of $\underline{\lambda}(\gamma^*_\varepsilon) = \varepsilon$, i.e.

$$\gamma^*_\varepsilon = 1-\varepsilon. \tag{17}$$

Further, if we get $\underline{\lambda}(\gamma) \leq \varepsilon$, for given (small) $\varepsilon > 0$ and $\gamma \in \langle\gamma^*_\varepsilon; \frac{4}{3}\rangle$, then can we compute $\delta_{\varepsilon,\gamma,\alpha}$, which is a solution of

$$F_\xi(\gamma a_{1-\frac{\alpha}{2}} + \delta_{\varepsilon,\gamma,\alpha}) - F_\xi(\gamma a_{\frac{\alpha}{2}} + \delta_{\varepsilon,\gamma,\alpha}) = 1-\alpha-\varepsilon. \tag{18}$$

Finally, we also get

$$\delta_{\varepsilon,\gamma,max} = \inf_{\alpha\in(0,1)} \delta_{\varepsilon,\gamma,\alpha} = \delta_{\varepsilon,\gamma,0}, \tag{19}$$

($\inf$ means the infimum operator). In particular, setting $\alpha = 0$ and for $\gamma \in \langle \gamma_\varepsilon^*; \frac{4}{3} \rangle$, $\gamma_\varepsilon^* = 1 - \varepsilon$, from (18) we have

$$F_\xi(\gamma\sqrt{3} + \delta_{\varepsilon,\gamma,max}) - F_\xi(-\gamma\sqrt{3} + \delta_{\varepsilon,\gamma,max}) = 1 - \varepsilon,$$

and we get the solution

$$\delta_{\varepsilon,\gamma,max} = \gamma\sqrt{3} - F_\xi^{-1}(1 - \varepsilon) = \sqrt{3}(\gamma + 2\varepsilon - 1). \tag{20}$$

Let X be rounded to order m such that using (10) the following holds true

$$|\Delta| = \frac{|X - X_*|}{\sigma_*}\gamma < 10^m \frac{5\gamma}{\sigma_*} < \delta_{\varepsilon,\gamma,max}. \tag{21}$$

In this case

$$10^m \frac{5\gamma}{\sigma_*} < \delta_{\varepsilon,\gamma,max},$$

and so, the proper order m of rounding X, for given (small) $\varepsilon > 0$ and for given $\gamma = \frac{\sigma_*}{\sigma}$, is

$$m < \log_{10} \frac{\sigma_* \delta_{\varepsilon,\gamma,max}}{5\gamma}. \tag{22}$$

In fact, for such m and for all $\alpha \in (0, 1)$ we get

$$\begin{aligned} 1 - \alpha_* &= P\left\{\gamma a_{\frac{\alpha}{2}} + \Delta < \xi < \gamma a_{1-\frac{\alpha}{2}} + \Delta\right\} \\ &\geq P\left\{\gamma a_{\frac{\alpha}{2}} + \delta_{\varepsilon,\gamma,max} < \xi < \gamma a_{1-\frac{\alpha}{2}} + \delta_{\varepsilon,\gamma,max}\right\} \geq 1 - \alpha - \varepsilon. \end{aligned}$$

In such a way we have obtained the ε-properly rounded result, where ${}_\varepsilon x$ is x rounded to the above mentioned proper order m and ${}_\varepsilon\sigma = \sigma_*$. Then the interval estimate

$$\langle {}_\varepsilon x - {}_\varepsilon\sigma a_{1-\frac{\alpha}{2}}; {}_\varepsilon x - {}_\varepsilon\sigma a_{\frac{\alpha}{2}} \rangle$$

is a realization of the interval estimator (12).

3. Conclusion

The ε-properly rounded result, under the assumption of a uniform distribution of the measured values, can be obtained by simple two-step procedure:

Step 1

For given (small and positive) ε and $\gamma = \sigma_*/\sigma$ compute the value $\delta_{\varepsilon,\gamma,max}$ according to (20).

(Note that for given ε the values $\delta_{\varepsilon,\gamma,max}$ are defined for γ greater than the threshold value γ_ε^* as defined in (17). Moreover, note that if $\gamma = 1$ we do not round the value of σ, i.e. $\sigma_* = \sigma$.)

Step 2

Round x to order m to get x_* where m is given by (22). So we get ${}_\varepsilon x = x_*$ and ${}_\varepsilon \sigma = \sigma_*$, i.e. the ε-properly rounded result ${}_\varepsilon x \pm {}_\varepsilon \sigma$ and also the ε-proper confidence interval estimate (13) for measured μ and arbitrary $\alpha \in (0,1)$.

4. Appendix

Lemma 1. *Let* $\xi \sim Un(-\sqrt{3}, \sqrt{3})$, $\alpha \in (0,1)$, $\gamma \in (\frac{2}{3}, 1\rangle$, *and* $\lambda(\gamma,\alpha,0) = P\left\{\gamma a_{\frac{\alpha}{2}} < \xi < \gamma a_{1-\frac{\alpha}{2}}\right\}$, *where* $a_{\frac{\alpha}{2}}$, $a_{1-\frac{\alpha}{2}}$ *are* $\frac{\alpha}{2}$ *and* $1-\frac{\alpha}{2}$ *quantiles, respectively, of the distribution of* ξ, *then*

$$\underline{\lambda}(\gamma) = \sup_{\alpha\in(0,1)} \{(1-\alpha) - \lambda(\gamma,\alpha,0)\} = 1 - \lambda(\gamma,0,0) = 1-\gamma. \tag{23}$$

PROOF. Using (5) and (6) the following relations hold true:

$$\begin{aligned}
\underline{\lambda}(\gamma) &= \sup_{\alpha\in(0,1)} \{(1-\alpha) - \lambda(\gamma,\alpha,0)\} \\
&= \sup_{\alpha\in(0,1)} \left\{(1-\alpha) - P\left\{\gamma a_{\frac{\alpha}{2}} < \xi < \gamma a_{1-\frac{\alpha}{2}}\right\}\right\} \\
&= \sup_{\alpha\in(0,1)} \left\{(1-\alpha) - \left[F_\xi(\gamma a_{1-\frac{\alpha}{2}}) - F_\xi(\gamma a_{\frac{\alpha}{2}})\right]\right\} \\
&= \sup_{\alpha\in(0,1)} \left\{(1-\alpha) - \left[\frac{\gamma(1-\alpha)+1}{2} - \frac{\gamma(\alpha-1)+1}{2}\right]\right\} \\
&= \sup_{\alpha\in(0,1)} \{(1-\alpha) - \gamma(1-\alpha)\} \\
&= \sup_{\alpha\in(0,1)} \{(1-\alpha)(1-\gamma)\} \\
&= 1-\gamma.
\end{aligned}$$

References

[1] *Guide to the Expression of Uncertainty in Measurement*, International Organization of Standardization, First edition, 1995.

[2] RAO, C.R. (1973). *Linear Statistical Inference and Its Applications*. Second Edition. Wiley, New York.

[3] WIMMER, G., WITKOVSKÝ, V., DUBY, T. (2000). Proper rounding of the measurement results under normality assumptions. *Measurement Science and Technology*, 11, 1659-1665.

[4] WIMMER, G., WITKOVSKÝ, V. (2002). Proper rounding of the measurement results under the assumption of triangular distribution. *Submitted to Measurement Science Review*, http://www.measurement.sk.