PROPER ROUNDING OF THE MEASUREMENT RESULTS UNDER THE ASSUMPTION OF TRAINGULAR DISTRIBUTION[1]

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Abstract

In this paper we propose general rules for proper rounding of the measurement results based on the method called ”ε-properly rounding” under triangular distribution assumptions.

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1. Introduction

The concept called ”ε-properly rounded result” was first introduced in [2] where the probability properties of the rounded measured values and their ε-proper confidence intervals were derived under normality assumptions on the distribution of the measurements.

Similarly, as in [2, 3] we will consider the result of the measurement given in the form

$$x \pm s, \tag{1}$$

where x is a realization of a random variable X with mean μ (the unknown measured quantity) and dispersion σ^2, s^2 is the estimate of dispersion σ^2. If σ, the standard deviation is known, instead of (1) we will use the notation $x \pm \sigma$. According to [1], σ is also called standard uncertainty.

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In this paper we will assume that the measurement errors follow a triangular distribution. Under this assumption we are interested in evaluation (from probabilistic point of view) of the effect of rounding of the measured values and their uncertainties. In particular, we are looking for such rounding of the measurement results (reported in the form (1)), that the approximation error caused by the rounding procedure leads at most to small, well defined and controlled deviation, say ε, from the nominal probability (significance level) if the standard statistical inference is applied.

Here we recall two different types of rounding (assuming standard decimal notation of numbers).

Definition 1. *We say that w_* is **rounded to $\mathbf{n}$ significant digits** (or **$\mathbf{n}$ significant digit rounding**) of the value w, where $n \in \{1, 2, \ldots\}$, if the following rules apply:*

1. *If the $(n+1)$-st digit in the decimal notation of w is $0, 1, 2, 3$ or 4 then the first n digits in the notation of w_* remain unchanged, and the remaining digits are zero or fall off.*
2. *If the $(n+1)$-st digit in the decimal notation of w is $5, 6, 7, 8$ or 9 then the n-th digit in the notation of w_* is increased by 1 and the remaining digits are zero or fall off.*

Example 1.

Rounding to 1 significant digit:

w	w_*
135.21	100
0.087	0.1
63.52	60

Rounding to 3 significant digits:

w	w_*
26832.632	26800
527.329	527
0.0852738	0.0853

Corollary 1. *If w is rounded to 1 significant digit then*

$$\frac{2}{3} < \frac{w_*}{w} \leq \frac{4}{3}. \tag{2}$$

If w is rounded to 2 significant digits then

$$\frac{20}{21} < \frac{w_*}{w} \leq \frac{22}{21}. \tag{3}$$

If w is rounded to 3 significant digits

$$\frac{200}{201} < \frac{w_*}{w} \leq \frac{202}{201}. \tag{4}$$

Definition 2. *Let* $\ldots d_2 10^2 + d_1 10^1 + d_0 10^0 + d_{-1} 10^{-1} + d_{-2} 10^{-2} \ldots$ *be the decimal expansion of* $|w|$. *We say that* w_* *is* ***rounded to order*** $\mathbf{m}$ *of the value* w*, where* $m \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$*, if the following rounding rules apply:*

1. *If the digit* d_m *in the decimal notation of* $|w|$ *is 0, 1, 2, 3 or 4 then all digits* d_n *of* $|w_*|$ *with* $n \leq m$ *are equal to zero or fall off and the remaining digits are unchanged.*

2. *If the digit* d_m *in the decimal notation of* $|w|$ *is 5, 6, 7, 8, or 9 then all digits* d_n *of* $|w_*|$ *with* $n \leq m$ *are equal to zero or fall off, the* d_{m+1} *digit is increased by one (with necessary carry over).*

Example 2.
The number $w = 3278.35876$ rounded to order -3 is $w_* = 3278.36$.
The number $w = 159.21$ rounded to order $+1$ is $w_* = 200$.
The number $w = 1121.85$ rounded to order $+3$ is $w_* = 0$.

Corollary 2. *If* w *is rounded to order* m *then*

$$|w_* - w| < 5 \times 10^m. \tag{5}$$

2. ε-properly rounded result (triangular distribution of the errors)

Random variable ξ has triangular distribution (ξ is triangularly distributed) on the interval $\langle a, b\rangle$, $(a < b)$, if its probability density function (pdf) is given by

$$f(x) = \begin{cases} \dfrac{2}{b-a} - \dfrac{2}{(b-a)^2}|a+b-2x|, & a \leq x \leq b \\ 0, & x \notin \langle a, b\rangle. \end{cases} \tag{6}$$

The random variable ξ with pdf (6) has its expectation and dispersion given by

$$\mathcal{E}(\xi) = \frac{a+b}{2} \qquad \mathcal{D}(\xi) = \frac{(b-a)^2}{24}.$$

Let x be a realized (observed) value of the random variable ξ, wich is triangularly distributed with its expectation (expected value) μ and standard uncertainty σ. Then

$$\zeta = \frac{\xi - \mu}{\sigma}$$

is a random variable with standardized triangular distribution, i.e. $\zeta \sim T(-\sqrt{6}, \sqrt{6})$.

The cumulative distribution function (cdf) of the random variable ζ is given by

$$F_T(z) = \begin{cases} 0, & \text{if } z < -\sqrt{6} \\ \frac{1}{2} + \frac{1}{\sqrt{6}}\left(z + \frac{z^2}{2\sqrt{6}}\right), & \text{if } -\sqrt{6} \le z < 0 \\ \frac{1}{2} + \frac{1}{\sqrt{6}}\left(z - \frac{z^2}{2\sqrt{6}}\right), & \text{if } 0 \le z < \sqrt{6} \\ 1, & \text{if } \sqrt{6} \le z. \end{cases} \tag{7}$$

The quantile function (inversion of the cdf function) of the standardized triangular distribution is defined for $\alpha \in (0,1)$ as

$$\tilde{F}_T(\alpha) = \begin{cases} \sqrt{6}(\sqrt{2\alpha} - 1), & \text{if } 0 \le \alpha < 0.5 \\ \sqrt{6}(1 - \sqrt{2(1-\alpha)}), & \text{if } 0.5 \le \alpha \le 1. \end{cases} \tag{8}$$

The α-quantile of the standardized triangular distribution will be denoted as $t(\alpha) = \tilde{F}_T(\alpha)$. Hence, for all $\alpha \in (0,1)$ we get

$$P\{t(\tfrac{\alpha}{2}) < \zeta < t(1 - \tfrac{\alpha}{2})\} = 1 - \alpha,$$

or

$$P\{\xi - \sigma t(1 - \tfrac{\alpha}{2}) < \mu < \xi - \sigma t(\tfrac{\alpha}{2})\} = 1 - \alpha.$$

Similarly, as in [3] we assume that the standard uncertainty σ is known. Let σ_* be the rounded value of σ (rounded to n significant digits). Denote

$$\gamma = \frac{\sigma_*}{\sigma}.$$

For the "approximately" standardized random variable

$$\zeta_* = \frac{\xi_* - \mu}{\sigma_*}$$

we get

$$\begin{aligned} 1 - \alpha_* &= P\left\{t(\tfrac{\alpha}{2}) < \frac{\xi_* - \mu}{\sigma_*} < t(1 - \tfrac{\alpha}{2})\right\} \\ &= P\left\{\gamma t(\tfrac{\alpha}{2}) + \frac{\xi - \xi_*}{\sigma_*} < \zeta < \gamma t(1 - \tfrac{\alpha}{2}) + \frac{\xi - \xi_*}{\sigma_*}\right\} \\ &= P\left\{\gamma t(\tfrac{\alpha}{2}) + \Delta < \zeta < \gamma t(1 - \tfrac{\alpha}{2}) + \Delta\right\}, \end{aligned} \tag{9}$$

where ζ be the standardized random variable with triangular distribution and further, $\Delta = \frac{\xi - \xi_*}{\sigma} = \frac{\xi - \xi_*}{\sigma_*} \times \frac{\sigma_*}{\sigma}$.

For selected values of $\varepsilon > 0$ and $\gamma \in \langle \gamma_\varepsilon^*, \frac{4}{3} \rangle$, (where γ_ε^* denote the "threshold" value of γ for given ε, given by (15)), in Table 1 we present the values of $\delta_{\varepsilon,\gamma,max} \ge 0$ which are to be used for determinig the proper order m for rounding of ξ,

$$m \le \left\lfloor \log_{10}\left(\frac{\sigma_* \delta_{\varepsilon,\gamma,max}}{5\gamma}\right) \right\rfloor, \tag{10}$$

where $\lfloor z \rfloor$ is the greatest integer less than or equal to z. For each fixed $\varepsilon > 0$ and $\gamma \in \langle \gamma_\varepsilon^*, \frac{4}{3} \rangle$, and for each $\alpha \in (0,1)$, it is ensured that

$$\begin{aligned} 1 - \alpha_* &= P\left\{ \gamma t(\tfrac{\alpha}{2}) + \Delta < \zeta < \gamma t(1 - \tfrac{\alpha}{2}) + \Delta \right\} \\ &\geq P\left\{ \gamma t(\tfrac{\alpha}{2}) + \delta_{\varepsilon,\gamma,max} < \zeta < \gamma t(1 - \tfrac{\alpha}{2}) + \delta_{\varepsilon,\gamma,max} \right\} \geq 1 - \alpha - \varepsilon. \end{aligned}$$

Let ${}_\varepsilon x$ be the rounded value of x (rounded to the order m, such that the inequality (10) holds true). Further, let ${}_\varepsilon \sigma = \sigma_*$ and let ${}_\varepsilon \xi$ be random variable with its realized (observed) value ${}_\varepsilon x$. Then we say, in accordance with Definition 3 in [2] or [3], that ${}_\varepsilon x \pm {}_\varepsilon \sigma$ is the *ε-properly rounded result of measurement*, and the random interval

$$\langle {}_\varepsilon \xi - {}_\varepsilon \sigma t(1 - \tfrac{\alpha}{2}); {}_\varepsilon \xi - {}_\varepsilon \sigma t(\tfrac{\alpha}{2}) \rangle \tag{11}$$

is the *ε-proper $(1-\alpha)$ confidence interval* for the measured (true) parameter μ, and the interval

$$\langle {}_\varepsilon x - {}_\varepsilon \sigma t(1 - \tfrac{\alpha}{2}); {}_\varepsilon x - {}_\varepsilon \sigma t(\tfrac{\alpha}{2}) \rangle \tag{12}$$

is its realized (observed) version.

For (small) fixed $\varepsilon > 0$ and for $\gamma \in \langle \gamma_\varepsilon^*, \frac{4}{3} \rangle$, the values of $\delta_{\varepsilon,\gamma,max} \geq 0$ (given in Table 1) were numerically calculated as

$$\delta_{\varepsilon,\gamma,max} = \inf_{\alpha \in (0,1)} \delta_{\varepsilon,\gamma,\alpha},$$

where $\delta_{\varepsilon,\gamma,\alpha} \geq 0$ (for each $\alpha \in (0,1)$) is the solution of the equation

$$(1 - \alpha - \varepsilon) - \left[F_T(\gamma t(1 - \tfrac{\alpha}{2}) + \delta_{\varepsilon,\gamma,\alpha}) - F_T(\gamma t(\tfrac{\alpha}{2}) + \delta_{\varepsilon,\gamma,\alpha}) \right] = 0.$$

Some interesting properties can be derived explicitely. Let us denote

$$\lambda(\gamma, \alpha, \delta) = P\left\{ \gamma t(\tfrac{\alpha}{2}) + \delta < \zeta < \gamma t(1 - \tfrac{\alpha}{2}) + \delta \right\}.$$

Then $\lambda(\gamma, \alpha, \delta)$ is non-increasing function of the parameter $\delta \geq 0$, and the maximum is reached at the value $\delta = 0$ (note, that the maximum can be reached at several different values of δ) and the following holds true

$$\lambda(\gamma, \alpha, 0) = F_T(\gamma t(1 - \tfrac{\alpha}{2})) - F_T(\gamma t(\tfrac{\alpha}{2})).$$

Moreover, we get

$$\lim_{\delta \to \infty} \lambda(\gamma, \alpha, \delta) = 0.$$

Now, let us define

$$\underline{\lambda}(\gamma) = \sup_{\alpha \in (0,1)} \left\{ (1 - \alpha) - \lambda(\gamma, \alpha, 0) \right\}.$$

Then the following Lemma holds true.

Lemma 1. *Let* $\zeta \sim T(-\sqrt{6}, \sqrt{6})$, $\alpha \in (0,1)$ *and* $\gamma \in (\frac{2}{3}; 1\rangle$. *Then*

$$\begin{aligned} \underline{\lambda}(\gamma) &= \sup_{\alpha \in (0,1)} \{(1-\alpha) - \lambda(\gamma, \alpha, 0)\} \\ &= (1-\alpha_{max}) - \lambda(\gamma, \alpha_{max}, 0) = \frac{1-\gamma}{1+\gamma}, \end{aligned} \tag{13}$$

where

$$\alpha_{max} = \left(\frac{\gamma}{1+\gamma}\right)^2. \tag{14}$$

Proof. Let us denote

$$\lambda(\gamma, \alpha) = (1-\alpha) - \lambda(\gamma, \alpha, 0).$$

With respect to (7) and (8), we get

$$\begin{aligned} \lambda(\gamma, \alpha) &= (1-\alpha) - \left[F_T(\gamma t(1 - \tfrac{\alpha}{2})) - F_T(\gamma t(\tfrac{\alpha}{2}))\right] \\ &= (1-\alpha) - 2\gamma(1-\sqrt{\alpha}) + \gamma^2(1-\sqrt{\alpha})^2. \end{aligned}$$

The maximum of the function $\lambda(\gamma, \alpha)$ is reached at the value α_{max}, which is given as the solution of the following equation

$$\frac{\partial \lambda(\gamma, \alpha)}{\partial \alpha} = (\gamma^2 - 1) + \frac{\gamma(1-\gamma)}{\sqrt{\alpha}} = 0.$$

From that we get

$$\alpha_{max} = \left(\frac{\gamma}{1+\gamma}\right)^2,$$

and further,

$$\begin{aligned} \underline{\lambda}(\gamma) &= \max_{\alpha \in (0,1)} \lambda(\gamma, \alpha) = \lambda(\gamma, \alpha_{max}) \\ &= (1-\alpha_{max}) - 2\gamma(1-\sqrt{\alpha_{max}}) + \gamma^2(1-\sqrt{\alpha_{max}})^2 = \frac{1-\gamma}{1+\gamma}. \end{aligned}$$

⋆

If, for chosen (small) $\varepsilon > 0$ and $\gamma \in (\frac{2}{3}; 1\rangle$, the inequality $\underline{\lambda}(\gamma) > \varepsilon$ holds true, then, according to the Definition 2, there does not exist the ε-properly rounded result of the measurement, even in the case if $\xi = \xi_*$ (i.e., in the case if we consider the precise result of measurement – x).

Based on that, for given $\varepsilon > 0$, we set γ_ε^* — the threshold value of the parameter γ, as the solution of the equation $\underline{\lambda}(\gamma_\varepsilon^*) = \varepsilon$, i.e. according to (13) as

$$\gamma_\varepsilon^* = \frac{1-\varepsilon}{1+\varepsilon}. \tag{15}$$

If $\underline{\lambda}(\gamma) \leq \varepsilon$, we can (for small value of $\varepsilon > 0$ and for $\gamma \in \langle\gamma_\varepsilon^*; \frac{4}{3}\rangle$) find such value of $\delta_{\varepsilon,\gamma,\alpha} \geq 0$ that

$$F_T(\gamma t(1 - \tfrac{\alpha}{2}) + \delta_{\varepsilon,\gamma,\alpha}) - F_T(\gamma t(\tfrac{\alpha}{2}) + \delta_{\varepsilon,\gamma,\alpha}) = 1 - \alpha - \varepsilon, \tag{16}$$

and moreover, the value $\delta_{\varepsilon,\gamma,max}$, defined by

$$\delta_{\varepsilon,\gamma,max} = \inf_{\alpha \in (0,1)} \delta_{\varepsilon,\gamma,\alpha}. \tag{17}$$

For given $\varepsilon \in (0; \frac{1}{4}\rangle$ and $\gamma \in \langle\gamma^*_\varepsilon; 1\rangle$ the explicite solution (17) is given as

$$\delta_{\varepsilon,\gamma,max} = \sqrt{6\left(\varepsilon - \frac{1-\gamma}{1+\gamma}\right)}. \tag{18}$$

3. Conclusion

The ε-properly rounded result of measurement (in the case of triangular distribution of the errors) can be derived by using the following simple two-steps method:

Step 1

For given small positive value of ε and for $\gamma = \sigma_*/\sigma$ derive the value of $\delta_{\varepsilon,\gamma,max}$ (either from the Table 1 or calculate it (for $\gamma \in \langle\gamma^*_\varepsilon; 1\rangle$) according to (18)). Table 1 seems to be superfluous, however for comparison reasons we present it in full form.

Note that, given small $\varepsilon > 0$, the values of $\delta_{\varepsilon,\gamma,max}$ are defined just for the values of the parameter γ wich are greater than the threshold value γ^*_ε, which is given by (15). If $\gamma = 1$, it means that the value σ was not rounded, i.e. $\sigma_* = \sigma$.

Step 2

Round the value of x to the order m (i.e. get the value x_*), where m is given by (10). Then set ${}_\varepsilon x = x_*$ and ${}_\varepsilon\sigma = \sigma_*$, and get the ε-proper result of measurement ${}_\varepsilon x \pm {}_\varepsilon\sigma$. Further, for arbitrary $\alpha \in (0,1)$, get according to (12) the realized ε-proper confidence interval for the measured (true) value μ.

Example 3. *Let $x = 127.835$, $\sigma = 15.287$. We round σ to 2 significant digits, i.e. $\sigma_* = 15, \gamma \doteq 0.98123$. For chosen $\varepsilon = 0.01$, we use (18) to calculate $\Delta_{0.01,0.98123,max} = 0.0561$. So according to (10) x ought to be rounded to order*

$$m \leq \left\lfloor \log_{10} \frac{15 \times 0.0561}{5 \times 0.98123} \right\rfloor = \lfloor -0.7659 \rfloor = -1.$$

So let $m = -1$. The 0.01-properly rounded result is 128 ± 15, the 0.01-proper 0.95-confidence interval estimate for the true value μ is $\langle 99.5, 156.5\rangle$.

Similarly, for chosen $\varepsilon = 0.05$, we get $\Delta_{0.05,0.98123,max} = 0.4931$, and x ought to be rounded to order

$$m \leq \left\lfloor \log_{10} \frac{15 \times 0.4931}{5 \times 0.98123} \right\rfloor = \lfloor 0.1783 \rfloor = 0.$$

So, we set $m = 0$. The 0.05-properly rounded result is 130 ± 15, the 0.05-proper 0.95-confidence interval estimate for the true value μ is $\langle 101.5, 158.5\rangle$.

4. Tables

Table 1. The values $\delta_{\varepsilon,\gamma,max}$. Triangular distribution.

	ε						
γ	0.005	0.01	0.02	0.03	0.04	0.05	0.1
0.81818	-	-	-	-	-	-	0.0000
0.82500	-	-	-	-	-	-	0.1570
0.85000	-	-	-	-	-	-	0.3369
0.87500	-	-	-	-	-	-	0.4472
0.90000	-	-	-	-	-	-	0.5331
0.90476	-	-	-	-	-	0.0000	0.5477
0.92308	-	-	-	-	0.0000	0.2449	0.6000
0.92500	-	-	-	-	0.0790	0.2574	0.6052
0.94175	-	-	-	0.0000	0.2449	0.3464	0.6481
0.95000	-	-	-	0.1617	0.2935	0.3823	0.6679
0.95238	-	-	-	0.1835	0.3060	0.3920	0.6735
0.95250	-	-	-	0.1845	0.3067	0.3925	0.6738
0.95500	-	-	-	0.2047	0.3192	0.4024	0.6796
0.95750	-	-	-	0.2230	0.3313	0.4120	0.6854
0.96000	-	-	-	0.2399	0.3429	0.4214	0.6910
0.96078	-	-	0.0000	0.2449	0.3464	0.4243	0.6928
0.96250	-	-	0.0731	0.2556	0.3540	0.4305	0.6967
0.96500	-	-	0.1146	0.2704	0.3649	0.4395	0.7022
0.96750	-	-	0.1445	0.2844	0.3753	0.4482	0.7077
0.97000	-	-	0.1692	0.2977	0.3855	0.4568	0.7132
0.97250	-	-	0.1907	0.3104	0.3954	0.4651	0.7186
0.97500	-	-	0.2099	0.3226	0.4050	0.4733	0.7239
0.97750	-	-	0.2275	0.3343	0.4144	0.4814	0.7292
0.98000	-	-	0.2437	0.3455	0.4236	0.4893	0.7344
0.98019	-	0.0000	0.2449	0.3464	0.4243	0.4899	0.7348
0.98250	-	0.0839	0.2589	0.3564	0.4325	0.4970	0.7396
0.98500	-	0.1211	0.2732	0.3669	0.4412	0.5046	0.7448
0.98750	-	0.1492	0.2868	0.3772	0.4497	0.5121	0.7498
0.99000	-	0.1728	0.2998	0.3871	0.4581	0.5195	0.7549
0.99005	0.0000	0.1732	0.3000	0.3873	0.4583	0.5196	0.7550
0.99250	0.0861	0.1934	0.3121	0.3968	0.4663	0.5267	0.7599
0.99500	0.1223	0.2120	0.3240	0.4062	0.4743	0.5338	0.7648

Table 1. The values $\delta_{\varepsilon,\gamma,max}$. Triangular distribution.

	ε						
γ	0.005	0.01	0.02	0.03	0.04	0.05	0.1
0.99502	0.1226	0.2122	0.3241	0.4063	0.4744	0.5339	0.7649
0.99525	0.1254	0.2138	0.3251	0.4071	0.4751	0.5345	0.7653
0.99550	0.1283	0.2156	0.3263	0.4080	0.4759	0.5352	0.7658
0.99575	0.1313	0.2173	0.3275	0.4089	0.4767	0.5359	0.7663
0.99600	0.1341	0.2190	0.3286	0.4099	0.4775	0.5366	0.7668
0.99625	0.1369	0.2207	0.3297	0.4108	0.4782	0.5373	0.7673
0.99650	0.1396	0.2224	0.3309	0.4117	0.4790	0.5380	0.7678
0.99675	0.1423	0.2241	0.3320	0.4126	0.4798	0.5387	0.7683
0.99700	0.1449	0.2258	0.3332	0.4135	0.4806	0.5394	0.7688
0.99725	0.1474	0.2275	0.3343	0.4144	0.4814	0.5401	0.7692
0.99750	0.1500	0.2291	0.3354	0.4153	0.4822	0.5408	0.7697
0.99775	0.1525	0.2308	0.3365	0.4162	0.4829	0.5415	0.7702
0.99800	0.1549	0.2324	0.3376	0.4171	0.4837	0.5422	0.7707
0.99825	0.1573	0.2340	0.3387	0.4180	0.4845	0.5429	0.7712
0.99850	0.1597	0.2356	0.3399	0.4189	0.4853	0.5436	0.7717
0.99875	0.1620	0.2372	0.3410	0.4198	0.4860	0.5443	0.7722
0.99900	0.1643	0.2388	0.3421	0.4207	0.4868	0.5450	0.7727
0.99925	0.1666	0.2403	0.3432	0.4216	0.4876	0.5457	0.7731
0.99950	0.1688	0.2419	0.3442	0.4225	0.4884	0.5464	0.7736
0.99975	0.1710	0.2434	0.3453	0.4234	0.4891	0.5470	0.7741
1.00000	0.1732	0.2450	0.3464	0.4243	0.4899	0.5477	0.7746
1.00025	0.1738	0.2455	0.3469	0.4248	0.4904	0.5482	0.7750
1.00050	0.1743	0.2461	0.3475	0.4253	0.4909	0.5487	0.7754
1.00075	0.1749	0.2466	0.3480	0.4258	0.4914	0.5491	0.7759
1.00100	0.1755	0.2472	0.3485	0.4263	0.4919	0.5496	0.7763
1.00125	0.1760	0.2477	0.3490	0.4268	0.4924	0.5501	0.7767
1.00150	0.1766	0.2483	0.3496	0.4273	0.4928	0.5506	0.7771
1.00175	0.1772	0.2488	0.3501	0.4278	0.4933	0.5510	0.7775
1.00200	0.1777	0.2494	0.3506	0.4283	0.4938	0.5515	0.7779
1.00225	0.1783	0.2499	0.3511	0.4288	0.4943	0.5520	0.7784
1.00250	0.1789	0.2505	0.3517	0.4293	0.4948	0.5525	0.7788
1.00275	0.1794	0.2510	0.3522	0.4298	0.4953	0.5529	0.7792

Table 1. The values $\delta_{\varepsilon,\gamma,max}$. Triangular distribution.

	ε						
γ	0.005	0.01	0.02	0.03	0.04	0.05	0.1
1.00300	0.1800	0.2516	0.3527	0.4303	0.4958	0.5534	0.7796
1.00325	0.1806	0.2521	0.3532	0.4308	0.4963	0.5539	0.7800
1.00350	0.1811	0.2526	0.3537	0.4313	0.4967	0.5544	0.7804
1.00375	0.1817	0.2532	0.3543	0.4318	0.4972	0.5548	0.7809
1.00400	0.1823	0.2537	0.3548	0.4323	0.4977	0.5553	0.7813
1.00425	0.1828	0.2543	0.3553	0.4328	0.4982	0.5558	0.7817
1.00450	0.1834	0.2548	0.3558	0.4333	0.4987	0.5562	0.7821
1.00475	0.1840	0.2554	0.3564	0.4338	0.4992	0.5567	0.7825
1.00498	0.1845	0.2559	0.3568	0.4343	0.4996	0.5571	0.7829
1.00500	0.1845	0.2559	0.3569	0.4344	0.4997	0.5572	0.7829
1.00750	0.1902	0.2614	0.3621	0.4393	0.5045	0.5619	0.7871
1.01000	0.1958	0.2668	0.3672	0.4443	0.5093	0.5666	0.7912
1.01250	0.2013	0.2722	0.3724	0.4493	0.5141	0.5712	0.7953
1.01500	0.2069	0.2775	0.3775	0.4542	0.5189	0.5758	0.7994
1.01750	0.2124	0.2829	0.3826	0.4591	0.5236	0.5804	0.8034
1.02000	0.2179	0.2882	0.3876	0.4640	0.5283	0.5850	0.8074
1.02250	0.2233	0.2935	0.3927	0.4688	0.5330	0.5896	0.8115
1.02500	0.2287	0.2987	0.3977	0.4737	0.5377	0.5941	0.8155
1.02750	0.2341	0.3040	0.4027	0.4785	0.5424	0.5986	0.8194
1.03000	0.2395	0.3092	0.4077	0.4833	0.5470	0.6031	0.8234
1.03250	0.2449	0.3143	0.4126	0.4880	0.5516	0.6076	0.8273
1.03500	0.2502	0.3195	0.4175	0.4928	0.5562	0.6120	0.8312
1.03750	0.2555	0.3246	0.4224	0.4975	0.5607	0.6164	0.8351
1.04000	0.2608	0.3297	0.4273	0.5022	0.5653	0.6209	0.8390
1.04250	0.2660	0.3348	0.4321	0.5068	0.5698	0.6253	0.8429
1.04500	0.2712	0.3399	0.4370	0.5115	0.5743	0.6296	0.8467
1.04750	0.2764	0.3449	0.4418	0.5161	0.5788	0.6340	0.8505
1.04762	0.2767	0.3452	0.4420	0.5163	0.5790	0.6342	0.8507
1.05000	0.2816	0.3499	0.4466	0.5207	0.5832	0.6383	0.8543
1.07500	0.3320	0.3988	0.4931	0.5656	0.6266	0.6804	0.8914
1.10000	0.3801	0.4454	0.5376	0.6084	0.6681	0.7206	0.9269
1.12500	0.4261	0.4899	0.5801	0.6493	0.7076	0.7590	0.9607

Table 1. The values $\delta_{\varepsilon,\gamma,max}$. Triangular distribution.

	ε						
γ	0.005	0.01	0.02	0.03	0.04	0.05	0.1
1.15000	0.4701	0.5325	0.6207	0.6884	0.7455	0.7958	0.9931
1.17500	0.5122	0.5733	0.6596	0.7259	0.7818	0.8310	1.0241
1.20000	0.5526	0.6124	0.6969	0.7618	0.8165	0.8647	1.0537
1.22500	0.5913	0.6499	0.7327	0.7963	0.8498	0.8970	1.0822
1.25000	0.6284	0.6859	0.7670	0.8293	0.8818	0.9281	1.1096
1.27500	0.6642	0.7204	0.8000	0.8611	0.9125	0.9579	1.1359
1.30000	0.6985	0.7537	0.8317	0.8916	0.9421	0.9866	1.1611
1.32500	0.7315	0.7857	0.8622	0.9210	0.9706	1.0142	1.1854
1.33333	0.7423	0.7961	0.8722	0.9306	0.9798	1.0232	1.1933

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