The Usage of the Knowledge of Convolution of Two Student's Distributions and Two Rectangular Distributions to Examine the Accuracy of Coverage Factor with the Method of Geometrical Sum

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Abstract. *The results of examining error evaluation of the coverage factor with approximate methods in indirect measurements have been presented in the paper. The characteristics of the convolution are compared to the known characteristics of the coverage factor for the convolution of two component distributions and of the coverage factor for the normal distribution. Comparison of the results obtained with the known evaluation of these errors in direct measurements enabled to determine the change tendency of the errors of coverage factor evaluation, when the number of components of standard uncertainties grows.*
The knowledge of coverage factor characteristics for the convolution of four selected probability distributions was used for the research.

Keywords: probability distribution, coverage factor, expanded uncertainty

1. Introduction

Each evaluation of the expanded uncertainty requires the choice of an approximate evaluation method of the coverage factor. In the methods suggested by the international document [1] it is necessary to decide whether the evaluated factor shall approach the factor for a normal distribution or for Student's distribution. Usually the sample size is the decisive factor in the choice. However, how the number of standard component uncertainties influences the choice of the evaluation method is unknown. The basis for estimating the accuracy of applied approximate method of the estimation of expanded uncertainty is the assumption on the necessity of determining the method, which could be regarded as the exact one.

An essentially appropriate concept was adopted, which is taken into consideration, that the method based on the command of the convolution of probability distributions of errors of components may be regarded as an exact method. Due to complexity and time-consuming character of computing the convolution of many distributions of components, the results of such computing are, in general, hardly ever published. Therefore, approximate methods are generally accepted and recommended.

There are the results of publications [2], [3], [5], [6] concerning the analysis of accuracy of approximate methods of expanded uncertainty estimation for simple direct measurement, when there are only two component standard uncertainties.

Indirect measurements are characterised usually by a larger number of standard components uncertainties.

In the present paper the analysis of accuracy of estimating the coverage factor in indirect measurements was described. The examination results for the convolution of two Student distributions and two rectangular distributions were described.

2. Characteristics of the convolution of two Student's distributions and two rectangular distrbutions

A measuring event, which utilizes a convolution of two Student distributions and two rectangular distributions is an example of indirect measurement carried out by means of two measuring devices, which in case of repeated measurements, show a scatter of results, and the number of measurements is small (n<30).

Therefore, four standard uncertainties are analyzed: two type-A standard uncertainties, which reflect a standard deviation of Student distribution and two type-B standard uncertainties, which reflect a standard deviation of rectangular distributions.

On the basis of the developed analytical description of coverage factors in case of the analyzed convolutions one is able to identify all parameters, which function are the factors. One is able to demonstrate that a coverage factor for the convolution S*S*R*R, from now on referred to as factor $k_{SSRR}(\alpha)$ is a function of 6 variables [4]: probability α, number of degrees of freedom $m_1 = n_1-1$ and $m_2 = n_2-1$ first and second Student's distributions and the ratio of standard uncertainties η_S, η_R and η (1):

$$k_{SSRR}(\alpha) = f(\alpha, m_1, m_2, \eta_S, \eta_R, \eta) \tag{1}$$

where:

- $\eta_S = \frac{u_{A_1}}{u_{A_2}}$ is the ratio of standard uncertainties of type A

- $\eta_R = \frac{u_{B_1}}{u_{B_2}}$ is the ratio of standard uncertainties of type B

- $\eta = \frac{u_A}{u_B} = \frac{\sqrt{u_{A_1}^2 + u_{A_2}^2}}{\sqrt{u_{B_1}^2 + u_{B_2}^2}}$ is the ratio of combined standard uncertainties of type A to type B.

3. Computational results of the coverage factor for the convolution

Calculations were executed for one probability value $\alpha = 0.99$, for small values m, and for the value η, ranging from 0.1 to 10.

Matlab program was used for the calculations and the following were assumed:

- approximation accuracy of the probability range α over the variable k, ε=1e-4,
- the number of integration ranges in the Simpson's method of integration 300,
- multiple j=20.

Computational results are presented in table 1.

Table 1. Values of the coverage factor $k_{SSRR}(\alpha)$

$1/\eta$	η	$m_1=3$ $m_2=3$	$m_1=9$ $m_2=9$
10		2,2531	2,2203
5		2,3875	2,2688
4		2,4812	2,3031
3		2,6688	2,3625
2		3,1141	2,4844
1	1	4,2031	2,7656
	2	5,1000	2,9906
	3	5,3625	3,0578
	4	5,4500	3,0800
	5	5,5188	3,0938
	10	5,5625	3,1125

Characteristics of the coverage factor are presented in the function of the ratio of standard uncertainties η and its converse. Characteristics of the coverage factor $k_{SSRR}(0.99)$ are compared to the characteristic of the coverage factor $k_{SR}(0.99)$ for the convolution S*R and the characteristic of the coverage factor $k_N(0.99)$ for a normal distribution.

Fig. 1 shows the characteristics of the coverage factor $k_{SSRR}(0.99)$ for $m_1=m_2=3$, $m_1=m_2=9$, $\eta_S=\eta_R=1$ in the function of the ratio of standard uncertainties $\eta=u_A/u_B$ and its converse.

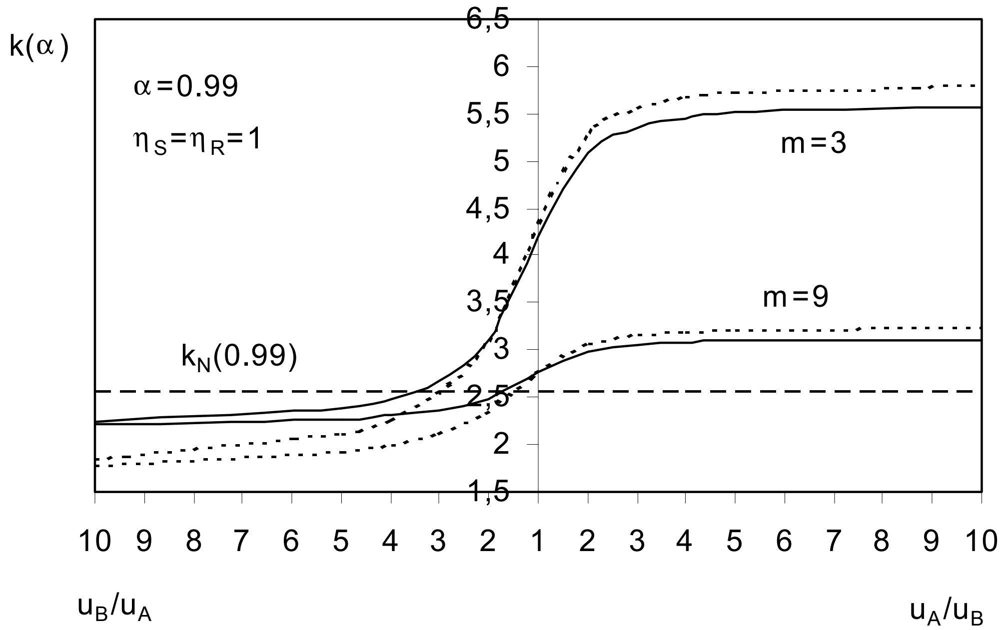

Fig. 1. Characteristics of the coverage factor $k_{SSRR}(0.99)$ for $m_1=m_2=3$, $m_1=m_2=9$, $k_{SR}(0.99)$ and $k_N(0.99)$ in the function of the ratio of standard uncertainties η and its converse.

In this situation both samples have the same number of degrees of freedom and none of the component standard uncertainties of type A and of type B is a domineering one. Broken line shows characteristics of the coverage factor $k_{SR}(0.99)$ for the convolution S*R, for $m = 3$ and $m = 9$, and the coverage factor $k_N(0.99)$ for a normal distribution.

In accordance with the central limit theorem, the characteristics of the coverage factor $k_{SSRR}(\alpha)$ and $k_{SR}(\alpha)$ clearly trend to approach the value of the factor $k_N(\alpha)$ as the sample size increases. The phenomenon is observed in the domain where $u_A>u_B$, further called domain A. Whereas in the domain where $u_B>u_A$ further called domain B, the influence of the sample size is much smaller and fades as the value of the ratio $u_B>u_A$ increases.

4. The method of geometrical sum

A method called the method of geometrical sum is utilized quite often in practical measurements. According to this method the expanded uncertainty is estimated as a geometrical sum of the component expanded uncertainties:

$$U = \sqrt{\sum_{j=1}^{N} u_j^2} \tag{2}$$

For indirect measurement the expanded uncertainty is calculated according to the relation presented below, taking into account that in the analyzed case of the convolution of four component distributions S*S*R*R, the expanded uncertainty will be equal to:

$$U = \sqrt{k_{S_1}^2(\alpha)\cdot u_{A_1}^2 + k_{S_2}^2(\alpha)\cdot u_{A_2}^2 + k_R^2(\alpha)\cdot\left(u_{B_1}^2 + u_{B_2}^2\right)} = k_{gSSRR}(\alpha)\cdot u_c \tag{3}$$

Assuming that all partial derivatives in the expression u_C are equal to one, the coverage factor estimated by means of a method of geometrical sum from now on referred to as $k_{gSSRR}(\alpha)$, will assume the form:

$$k_{gSSRR} = \frac{\sqrt{k_{S_1}^2(\alpha)\cdot u_{A_1}^2 + k_{S_2}^2(\alpha)\cdot u_{A_2}^2 + k_R^2(\alpha)\cdot\left(u_{B_1}^2 + u_{B_2}^2\right)}}{\sqrt{u_{A_1}^2 + u_{A_2}^2 + u_{B_1}^2 + u_{B_2}^2}} \tag{4}$$

After appropriate substitutions, factor $k_{gSSRR}(\alpha)$ expressed as a function of standard uncertainties ratios η and η_S and specific of coverage factors, will be produced:

$$k_{gSSRR}(\alpha) = \sqrt{\frac{\eta^2\left(k_{S_1}^2(\alpha)\cdot\eta_S^2 + k_{S_2}^2(\alpha)\right) + \left(\eta_S^2+1\right)\cdot k_R^2(\alpha)}{\left(\eta^2+1\right)\left(\eta_S^2+1\right)}} \tag{5}$$

Fig. 2 presents characteristics of coverage factor $k_{gSSRR}(0.99)$ and $k_{SSRR}(0.99)$ in the function of the ratio of standard uncertainties η and its converse.

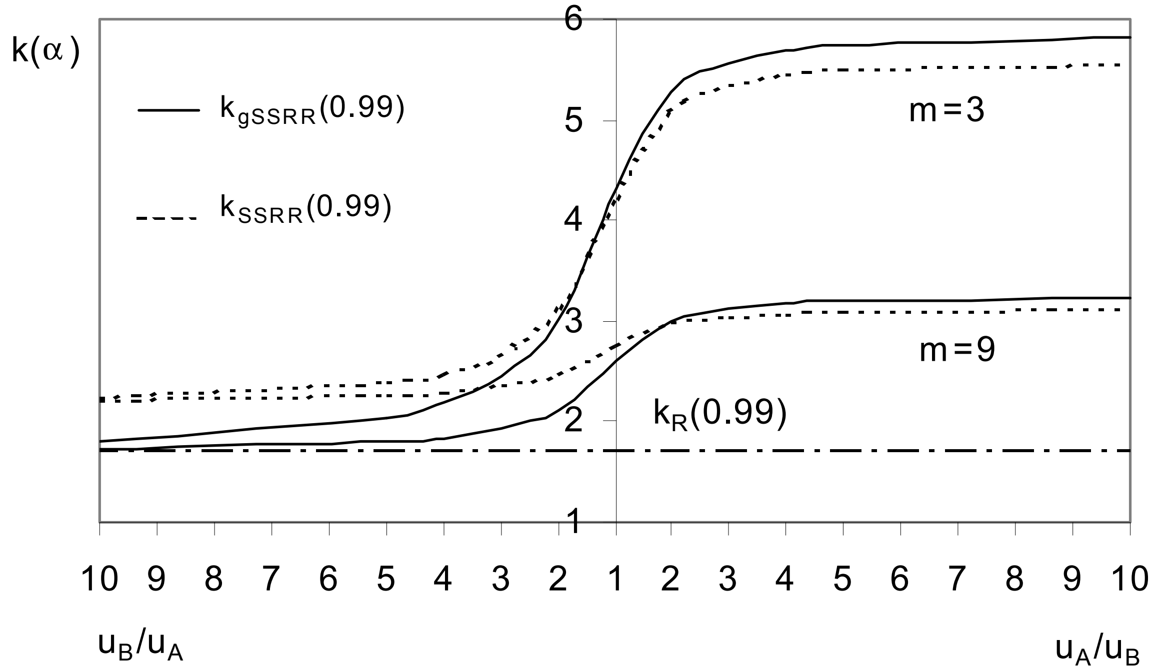

Fig. 2. Characteristics of the coverage factor $k_{gSSRR}(0.99)$, $k_{SSRR}(0.99)$ and $k_R(0.99)$ in the function of the ratio of standard uncertainties η and its converse.

A characteristic feature of the computed factor $k_{gSSRR}(0.99)$ is that in domain A its values will not differ significantly from the values of factor $k_{SSRR}(0.99)$, which are exact values. With the increasing number of degrees of freedom m, the differences diminish. In domain B the values of the analyzed factor only to a lesser extent depend on the number of degrees of freedom and the values of factors $k_{gSSRR}(0.99)$ are getting close to the values of factor $k_R(0.99)$ for a rectangular distribution.

According to the assumption that the value of coverage factor for the analyzed convolution of component distributions may be regarded as an exact value, the absolute value of error estimation by means of this approximate method is defined as:

$$\delta = \frac{\left| k_{gSSRR}(0.99) - k_{SSRR}(0.99) \right|}{k_{SSRR}(0.99)} \tag{6}$$

Fig. 3 presents the absolute error values δ of factor estimations $k_{gSSRR}(0.99)$ against errors δ' of the factor estimation $k_{gSR}(0.99)$ in the function of the ratio of standard uncertainties η and its converse for various values of m, where:

$$\delta' = \frac{\left| k_{gSR}(0.99) - k_{SR}(0.99) \right|}{k_{SR}(0.99)} \tag{7}$$

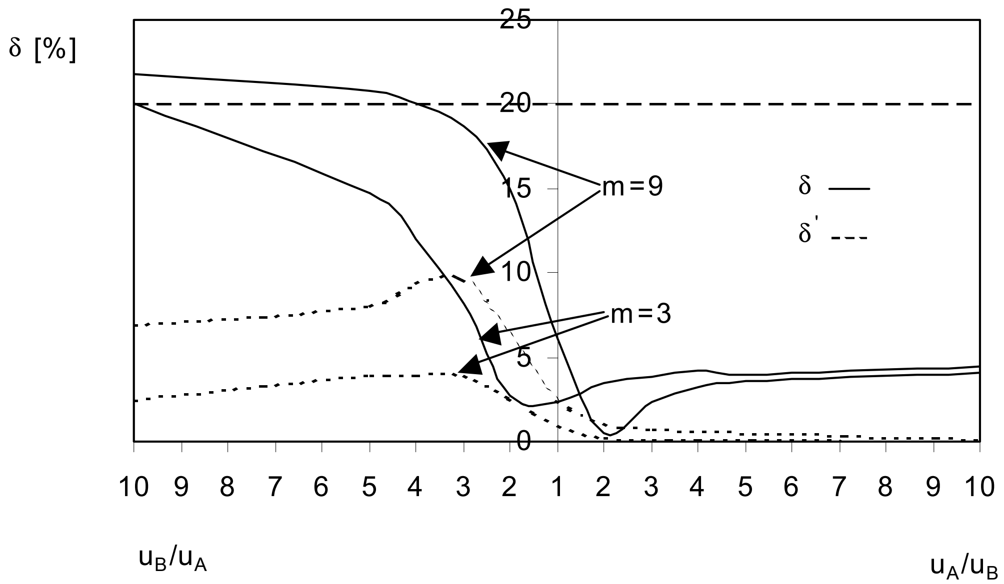

Fig. 3. Absolute error values δ of factor estimations $k_{gSSRR}(0.99)$ and error values δ' of the factor estimation $k_{gSR}(0.99)$ in the function of the ratio of standard uncertainties η and its converse

5. Conclusion

According to Fig. 3 there is a limitation of utilization of the method of geometrical sum especially in the domain where the ratio u_B/u_A increases. In spite of a relatively big increase of errors in case of indirect measurements in comparison with errors in direct measurements, they exceed the assumed value 20% in a small range of changes of standard uncertainties ratio for the number of degrees of freedom $m = 9$.

The present results of research, which concern the trend of changes of errors of coverage factor estimations are connected with characteristics features of coverage factors $k_{gSSRR}(0.99)$ estimated by means of the method of geometrical sum and the characteristics of the coverage factor $k_{SSRR}(0.99)$ for the analysed convolution.

References

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