Calculation of the approximate coverage factor value for the convolution of two Student's distributions and one rectangular distribution

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Abstract. *The result of examining error evaluation of the coverage factor in indirect measurement has been presented in the paper. To determine coverage factor value, the use of mathematical model, for analysed convolution of component distributions, has been presented. The knowledge of coverage factor characteristics for the convolution of two Student's distributions and one rectangular distribution was used for the examination.*

Keywords: expanded uncertainty, mathematical model, coverage factor

1. Introduction

Each evaluation of the expanded uncertainty *U* requires the choice of an approximate evaluation method of the coverage factor. In the methods suggested by the international document [1] it is necessary to decide whether the evaluated factor shall approach the factor for a normal distribution or for Student's distribution. Usually the sample size is the decisive factor in the choice. However, how the number of standard component uncertainties influences the choice of the evaluation method is unknown. The basis for estimating the accuracy of applied approximate methods of the estimation of expanded uncertainty is the assumption on the necessity the assessment methods, which could be regarded as exact one. An essentially appropriate concept was adopted, that the method based on the command of the convolution of probability distributions of errors of components may be regarded as an exact method. Due to complexity and time-consuming character of computing the convolution of many distributions of components, the results of such computing are, in general, hardly ever published. Therefore, approximate methods are generally accepted and recommended.

The present paper continues research in tendencies of changes in error evaluation of the coverage factor $k(\alpha)$ by means of approximate methods with an increasing number of component uncertainties. Results for a convolution of two Student's distributions and one rectangular distribution are presented in this paper.

2. Characteristics of the convolution of two Student's distributions and one rectangular distribution

A measuring event, which utilizes a convolution of two Student distributions and one rectangular distribution is an example of indirect measurement carried out by means of two measuring devices, which, in case of repeated measurements, show a scatter of results, a type-B standard uncertainty of one of the devices can be neglected, and the number of measurements is small ($n < 30$). Therefore, three standard uncertainties are analyzed: type-B standard uncertainty, which reflects a standard deviation of rectangular distribution and two type-A standard uncertainties, which reflect a standard deviation of Student distribution.

On the basis of the developed analytical description of coverage factors in case of the analyzed convolutions one is able to identify all parameters, which function are the factors. One is able to demonstrate that a coverage factor for the convolution S*S*R, from now on referred to as factor $k_{SSR}(\alpha)$ is a function of 5 variables [4]: probability α, number of degrees of freedom $m_1 = n_1$-1 and $m_2 = n_2$-1 first and second Student's distributions and the ratio of standard uncertainties η_S, and η (1):

$$k_{SSR}(\alpha) = f(\alpha, \eta, \eta_S, m_1, m_2). \tag{1}$$

Symbol suggested by the international document [1] shall be used for further consideration. According to the document:

$\eta_S = \dfrac{u_{A_1}}{u_{A_2}}$ is the ratio of standard uncertainties of type A,

$\eta = \dfrac{u_{cA}}{u_B} = \dfrac{u_A}{u_B}$ is the ratio of combined standard uncertainties of type A to type B.

3. Computational results of the coverage factor for the convolution

Calculations were executed for one selected probability value $\alpha = 0.95$, for small values m, and for the value η, ranging from 0.1 to 10. Matlab program was used for the calculations.

Computational results are presented in table 1.

Table 1. Values of the coverage factor $k_{SSR}(0.95)$

1/η	η	m_1=3	m_1=6	m_1=9
		m_2=3	m_2=6	m_2=9
10		1.6754	1.6593	1.6561
5		1.7650	1.7137	1.7023
4		1.8211	1.7477	1.7313
3		1.9184	1.8038	1.7797
2		2.1313	1.9201	1.8761
1	1	2.6195	2.1619	2.0617
	2	3.0109	2.3363	2.1844
	3	3.1297	2.3881	2.2175
	4	3.1781	2.4085	2.2306
	5	3.1988	2.4188	2.2375
	10	3.2324	2.4313	2.2453

Fig.1 present hypothetical characteristics of coverage factor $k_{SSR}(0.95)$ for m_1=m_2=3, m_1=m_2=6 and for m_1=m_2=9, $\eta_S = 1$, in the function of the ratio of standard uncertainties $\eta = u_A / u_B$ and its converse. In this situation both samples have the same number of degrees of freedom and none of the component standard uncertainties of type A and of type B is a domineering one. Broken line show characteristics of the coverage factor $k_N(0.95)$ for a normal distribution.

In accordance with the central limit theorem, the characteristics of the coverage factor $k_{SSR}(\alpha)$ clearly trend to approach the value of the factor $k_N(\alpha)$ as the sample size increases. The phenomenon is observed in the domain where $u_A > u_B$, further called domain A. Whereas in the domain where $u_B > u_A$ further called domain B, the influence of the sample size is much smaller and fades as the value of the ratio u_B/u_A increases.

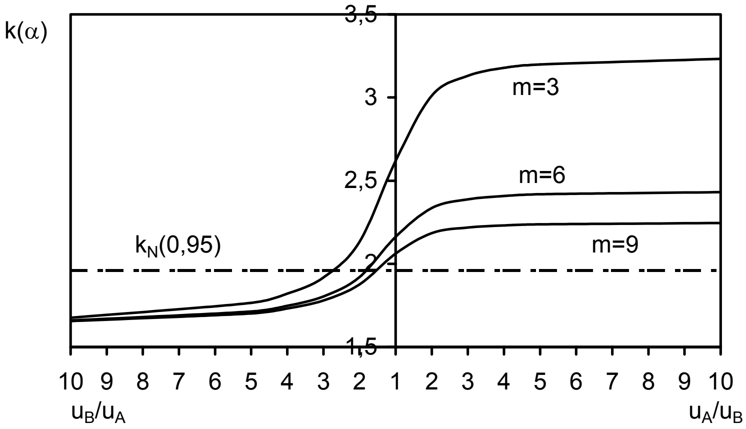

Fig. 1. Characteristics of coverage factor $k_{SSR}(0.95)$ and $k_N(0.95)$ in the function of the ratio of standard uncertainties $\eta = u_A/u_B$ and its converse.

In accordance with the assumption that the value of the coverage factor for a tested convolution of distributions of components can be treated as a precise value, the absolute value of evaluation error with the approximate method shall be defined as:

$$\delta = \frac{|\, k(\alpha) - k_{SSR}(\alpha)\,|}{k_{SSR}(\alpha)}, \tag{2}$$

where $k(\alpha)$ is the coverage factor evaluated by means of the approximate method, whereas $k_{SSR}(\alpha)$ is a ratio determined on the basis of the known convolution of distributions of components.

According to the conducted analysis, the application of approximate methods of evaluation of expanded uncertainty in many cases results in exceeding the assumed value of evaluation error coming from equation (2), equal to 20% [2, 3, 4]. Therefore, attempts were made to improve precision of the approximate method by creating mathematical models describing coverage factors for the analyzed convolutions of distributions of components. There exist several ways of describing the tested sample curves presented in Fig. 1. If a particular standard uncertainty, of type A or type B, prevails, the value of the coverage factor converges appropriately with the value of coverage factor $k_N(\alpha)$ for a normal distribution or $k_R(\alpha)$ for a rectangular distribution. A measuring situation, where the two uncertainties (type A and B) are comparable, is of particular importance. Therefore, simulation of the coverage factor $k_{SSR}(\alpha)$ was made concerning changes in the value of the ratio of standard uncertainty $\eta = u_A/u_B$ from 0.5 to 2. The equation describing the model $k_m(\alpha)$ of the coverage factor $k_{SSR}(\alpha)$ is the relationship:

$$k_m(\alpha) = a + bx. \tag{3}$$

Numeric values of parameters a and b were determined taking advantage of the least square method. Simulation was conducted for one selected probability values $\alpha = 0.95$ and for various numbers of degrees of freedom m. On the basis of obtained results, the value of simulation was determined, calculated in accordance with the relationship (2), where coverage factor $k(\alpha)$ assumes values $k_m(\alpha)$.

Fig. 2 present hypothetical characteristics of absolute error values δ of factor estimations $k_m(\alpha)$ for probability $\alpha = 0.95$ and number of degrees of freedom $m = 3$, $m = 6$ and $m = 9$.

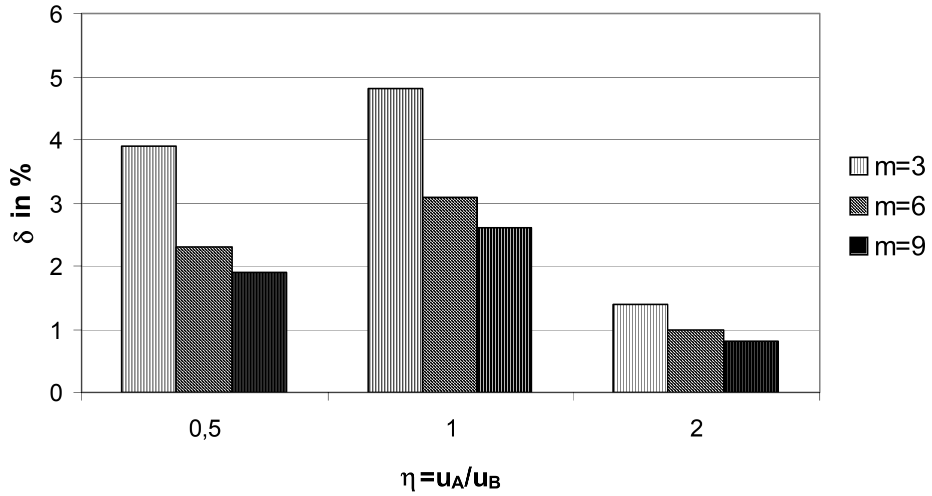

Fig. 2. Absolute error values δ of factor estimation $k_m(0.95)$ in the function of the ratio of standard uncertainties $\eta = u_A/u_B$ and its converse.

4. Conclusions

The determined simulation error values rise with a decrease in the number of degrees of freedom m. The maximum value of an error for the analyzed probability α and the number of degrees of freedom m does not exceed 5%, which constitutes considerable improvement in precision $k(\alpha)$ evaluation over the values obtained by means of the approximate methods presented in papers [2, 3, 4]. The developed model descriptions have considerably reduced the error of approximate evaluation of expanded uncertainty estimation for a common situation where a convolution of two Student's distributions and one rectangular distribution occurs.

References

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