One-sided Tolerance Factors of Normal Distributions with unknown mean and variability

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Abstract. In the contribution are given some approximate formulas as well as a derivation of exact formula for computing of the one-sided tolerance factors of normal distribution with unknown mean and unknown variability. It is also shown that the exact computation of the factors is closely related with non-central t-distributions.

Keywords: tolerance intervals, mean, variability

1. Introduction

In [1] we deal with computation of tolerance factors for the two-sided tolerance limits of a normal distribution with unknown parameters. Results are published in [2]. In [3] are given the results for more than one normal distribution with different means and common variability. All the above mentioned parameters are unknown. In this contribution we are interested in one-sided tolerance limits of a normal distribution with unknown parameters.

2. One-sided tolerance intervals

Let random sample $X_1, X_2, \ldots, X_n$ be taken from distribution $N(\mu, \sigma^2)$. Parameters $\mu$ and $\sigma^2$ are unknown.

We will find intervals, which with confidence $1 - \alpha$ ($0 < \alpha < 1$) cover at least the fraction $p$ ($0 < p < 1$) of values of the distribution $N(\mu, \sigma^2)$. Such intervals are called 100 $p$% tolerance intervals. In the following text we are only dealing with one-sided tolerance intervals which can be right-hand or left-hand.

For right-hand tolerance interval $(-\infty, \bar{X} + kS)$ is valid

$$P[P(X < \bar{X} + kS) \geq p] = 1 - \alpha$$

(1)

and for left-hand tolerance interval $(\bar{X} - kS, \infty)$ is valid

$$P[P(X > \bar{X} - kS) \geq p] = 1 - \alpha$$

(2)

where $X$ is a normally distributed variable with mean $\mu$ and variability $\sigma^2$ from which the random sample $X_1, X_2, \ldots, X_n$ was taken, $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$ and $S^2 = (n-1)^{-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$. 

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The value of the factor $k$ is determined so that the tolerance intervals with the confidence $1 - \alpha$ cover at least fraction $p$ of the values of the distribution $N(\mu, \sigma^2)$. Number $1 - \alpha$ named the confidence level and constant $k$ is the one-sided tolerance factor.

3. Non-central $t$-distribution

Let us consider random variables

$$ Y = \frac{X - \mu}{\sigma} \sqrt{n}, \quad V = \frac{\nu S^2}{\sigma^2}, \quad T = \frac{Y + \delta}{\sqrt{V}} $$

The random variable $Y$ has a standard normal distribution $N(0,1)$. It is independent of the random variable $V$, which has $\chi^2$-distribution with $\nu = n - 1$ degrees of freedom. The random variable $T$ has a non-central $t$-distribution with $\nu$ degrees of freedom and non-centrality parameter $\delta$ ($-\infty < \delta < \infty$), which is denoted by $T \sim T(\nu, \delta)$. Its probability density function is

$$ f_\nu(t, \delta) = \frac{\nu^{\nu/2} \exp\left(-\frac{\nu}{2} \delta^2 / (\nu + t^2)\right)}{2^{\nu/2} \Gamma(\nu/2) \sqrt{\pi \nu} \left(1 + \frac{t^2}{\nu}\right)^{\nu/2}} \int_0^\infty y^{\nu-1} \exp\left(-\frac{\nu}{2} \delta^2 / (y^2 + t^2)\right) dy, \quad -\infty < t < \infty $$

where $\Gamma(\nu) = \int_0^\infty x^{\nu-1} \exp(-x) dx$ is the gamma function ($\nu > 0$).

Let us denote the $\alpha$th quantile ($0 < \alpha < 1$) of the distribution $T(\nu, \delta)$ by $t_\alpha(\nu, \delta)$. It follows from (4) that for any real numbers $t$ and $\delta$ holds

$$ t_\alpha(\nu, \delta) = -t_{1-\alpha}(\nu, -\delta) $$

The most widespread tables of percentiles of non-central $t$-distributions can be found in the monograph [5], where they are computed with accuracy to five decimal places for $\alpha = 0.01; 0.025; 0.05; 0.10; 0.20; 0.30; 0.70; 0.80; 0.90; 0.95; 0.975; 0.99$, $\nu = 1(1)60$ and $\delta = 0.1(1)8.0$.

4. Approximate computation of one-sided tolerance factors

There exist a great number of approximations of the one-sided tolerance factors [7] and [8]. The Wallis’s approximation [9] is the best known and up till now the most used. It was derived on the basis of the approximation of the statistic $\overline{X} + kS$ by the normal distribution $N\left(\mu + k\sigma, \frac{\sigma^2}{n} + \frac{\sigma^2 k^2}{2n - 2}\right)$. The one-sided tolerance factor $k$ is given by the relationship

$$ k \approx \frac{u_p + \sqrt{u_p^2 - AB}}{A} $$
where \( A = 1 - \frac{u_{1-a}^2}{2(n-1)} \), \( B = u_p^2 - \frac{u_{1-a}^2}{n} \) and \( u_p \), \( u_{1-a} \) are quantiles of \( N(0,1) \). Slightly better results are given by Jennett’s and Welch’s approximation [10] by means of the quantiles of the non-central \( t \)-distribution

\[
 t_a(v, \delta) \approx \frac{\delta b_v + u_a \sqrt{b_v^2 + (1-b_v^2)\left(\delta^2 - u_a^2\right)}}{b_v^2 - u_a^2(1-b_v^2)} \tag{7}
\]

where \( b_v = (\Gamma((v+1)/2)/\Gamma(v/2)) \times \sqrt{2/v} \) For a small \( \delta \) is convenient the approximation by van Eeden [8], which was derived by means of Cornish-Fisher expansion [11] \( (v = n - 1) \)

\[
 t_a(v, \delta) \approx u_a + \frac{u_a^3 + u_a}{4v} + \frac{5u_a^5 + 16u_a^3 + 3u_a}{96v^2} + \frac{2u_a^3 + 12u_a^2 + 1}{32v^2} + \frac{u_a^3 + 4u_a^2}{16v^2} \delta - \frac{u_a^2 - 1}{24v^2} \delta^2 - \frac{u_a}{32v^2} \delta^3 \tag{8}
\]

Neither of the above mentioned approximations can be considered to give good results in general. This non-availability does not contain Akahira’s approximation [12]. The quantiles of the non-central \( t \)-distribution \( x = t_a(v, \delta) \) can be found by solving the following equation

\[
 \frac{b_v x - \delta}{\sqrt{1 + x^2(1-b_v^2)}} = u_a - \frac{x^3(u_a^2 - 1)}{24\sqrt{1 + x^2(1-b_v^2)^3}} \left( 1 + \frac{1}{4v^2} \right) \tag{9}
\]

where \( b_v \) is the same as given in (7) and \( u_a \) is the \( a \)th quantile of the standard normal distribution. The approximation (9) yields good results even if \( v \) is small. When \( v \geq 200 \) there are many cases where the error is less than \( 10^{-4} \). There is a disadvantage that the equation (9) can be solved for unknown \( x \) only by using numerical methods.

### 5. Exact computation of tolerance factors

For the right-sided tolerance interval \( (-\infty, \bar{X} + kS) \) is

\[
 Z = P(X < \bar{X} + kS) = \Phi\left( \frac{\bar{X} + kS - \mu}{\sigma} \right) \tag{10}
\]

where \( \Phi \) is the distribution function of a standard normal distribution \( N(0,1) \). For given \( p \) \( (0 < p < 1) \) the inequality \( Z \geq p \) is equivalent to

\[
 \frac{\bar{X} + kS - \mu}{\sigma} \geq u_p \tag{11}
\]

where \( u_p \) is the \( p \)th quantile of \( N(0,1) \). The tolerance factor \( k \) is determined so that for the given confidence coefficient \( 1 - \alpha \) is fulfilled the condition \( P(Z \geq p) = 1 - \alpha \). By using consecutive modifications and utilizing (3) we obtain
The random value \( T = \frac{\bar{X} - \mu - \sigma u_p}{S} \sqrt{n} \) has a distribution \( T(n-1,-u_p\sqrt{n}) \). From relations (5) and (12) it follows that

\[
k = -\frac{t_{a}(n-1,-u_p\sqrt{n})}{\sqrt{n}} = t_{1-a}(n-1,u_p\sqrt{n})
\]  

(13)

For the left-sided tolerance interval \( (\bar{X} - kS, \infty) \) we will also get \( P(T \leq k\sqrt{n})=1-\alpha \), where \( T \) has a distribution \( T(n-1,u_p\sqrt{n}) \) and \( k \) is given by (13) as well.

6. Conclusion

The exact computation of the one-sided tolerance factors \( k \) is closely related with the exact computation of quantiles of non-central \( t \)-distribution. One-sided tolerance intervals can be successfully applied in statistical quality control. For those who are interested in the mentioned field we recommend [13], [14].

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References


