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Practical Procedure for Position Tolerance Uncertainty Determination via Monte-Carlo Error Propagation

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Determination of realistic uncertainty values in coordinate metrology is a challenging task due to the complexity of the implementation of numerical algorithms involved. Monte-Carlo error propagation is used to estimate the uncertainty of a position tolerance using least-squares criterion. In this paper all the required steps are sequentially performed using a number real-world datasets. Since no reference data sets are available for position tolerance evaluation hence drawings and numerical values of such data sets are proposed.

Keywords: Uncertainty evaluation, propagation of distributions, Monte-Carlo simulation, position tolerance

> *"So, a result without reliability (uncertainty) statement cannot be published or communicated because it is not (yet) a result. I am appealing to my colleagues of all analytical journals not to accept papers anymore which do not respect this simple logic". (Paul De Bièvre, [1])*

1. Introduction

Measurement result besides a measured value should contain a measurement uncertainty. Determination of realistic uncertainty values is a challenging task in coordinate metrology. It is caused by the fact that coordinate measuring machines (CMM) are very versitile instrument that is affected by vast number of factors.

According to the ISO Guide to the Expression of Uncertainty in Measurement [2] a measuring task should have a mathematical model describing how the measured value depends on input quantities. Using such a model one can aquire probability distribution of the measured value. Standard deviation of this distribution can be used as an uncertainty of the measured value. As is shown by Cox et al. [3] in most cases it is enough to know mean value and standard deviation and there is no need to deal with explicit probability distributions.

Monte-Carlo error propagation is widely used for this purpose [4]. The procedure is known as the "virtual measuring machine" (VCMM) [5]. This method is used in certified calibration laboratories in Germany, UK and other contries [6] to maintain traceability [7] of the measurement results.

Monte-Carlo simulation assumes that a probability distribution of the input data is known. Usualy a normal distribution is used, but in principle any type of distribution can be considered within this method. For an input data set one generates random deviations (simulating uncertainty in the input data) and then a measurement model is computed. To get the probability distribution of the result one should repeat the process. The process stops when the running average stabilizes. This value (the running average) is the measurement result, and its standard deviation is the standard uncertainty (according to [2]).

In this paper Monte-Carlo error propagation is used to estimate the uncertainty of a position tolerance [8] using least-squares fit[1].

2. Related works

Dhanish and Mathew [10] used Monte-Carlo error propagation do determine the effect of CMM point coordinate uncertainty on uncertainties of circular features. Liu et al. [11] utilized Monte-Carlo error propagation to estimate form deviation uncertainties. Romano and Vicario [12] proposed a research of an influence of position tolerance uncertainty on parts acceptance using Monte-Carlo simulation. Sworn et al. [13] considered how optimization criteria (Gauss, Chebishev, inscribed and circumscribed circles) affects measurement results. Hopp [14] proposed analytic method to evaluate the uncertainty a three-point circle fitting. Horn studied the absolute orientation problem and proposed closed-form solutions using unit quaternion [16] and orthonormal matrices [17], his evaluation of a best-fit transformation is used in this paper.

3. Position tolerance, least-squares fit and absolute orientation

The goal of position tolerance is to ensure assemblability of workpieces. Fitting of workpieces can be viewed as an optimization problem [18]: find the the parameters to transform one coordinate system into the other that optimize a particular fitting objective for a set of points. Hopp claims [18] that averaging fits used in metrology are biased with respect

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[1]Though a median-polish fit [9] is also used to evaluate position tolerance is was beyond the scope of this paper.

to the extremal fit objectives suggested by tolerancing theory and the averaging fits will be different from extremal fits[2]. In this paper we use the averaging least-squares fit because of its wide use in CMM software.

Position tolerance with a least-squares fit is a case of planar absolute orientation problem — finding the relationship between two planar coordinate systems of nominal and actual data respectively. Suppose we have a set of N nominal points in $\mathbb{R}^2$ and a set of N measured points in $\mathbb{R}^2$. The absolute orientation problem is to determine an affine transformation (rigid motion [19]) such that each measured point is moved near the corresponding nominal point minimizing the specified criterion by a numerical process, Gauss or Levenberg—Marquardt optimization or some other.

4. Uncertainty evaluation

According to [2] in most cases a measurand Y is not measured directly, but is determined from N other quantities $X_1, X_2, \ldots X_N$ through a functional relationship: $Y = f(X_1, X_2, \ldots X_N)$.

The combined standard uncertainty of the function f can be evaluated as:

$$u(Y) = \sqrt{\sum_{i=1}^{N} \left(\frac{\partial f}{\partial X_i}\right)^2 \cdot u^2(X_i)}.$$

The function f corresponds to the measurement process and the method of the evaluation. Depending on how the standard uncertainty is evaluated, the input quantities $X_1, X_2, \ldots X_N$ may be grouped into [2]:

- **Type A evaluation (of uncertainty)** — method of evaluation of uncertainty by the statistical analysis of series of observations.
- **Type B evaluation (of uncertainty)** — method of evaluation of uncertainty by means other than statistical analysis of series of observations (i.e. previous measurement data, manufacturer's specifications, handbooks).

Most of CMMs measure volumetric coordinates $(x; y; z)$ of the probe center. It is possible [20] to consider all CMMs errors as the uncertainties of $(x; y; z)$-coordinates' measurement.

The result of a position tolerance evaluation is a set of deviations for each feature. Every deviation has an uncertainty. Metrological vocabulary [7] defines measurement uncertainty as a "parameter, associated with the result of measurement, that characterizes the dispersion of the values". The limits of the distribution for a specific confidence level are called expanded uncertainty [2]. To interpret the results presented in this paper let us assume that

- the deviations of points coordinates are in spherical coordinate system with polar radius r and polar angles θ and ϕ;
- the radius deviation is random, independent, normally distributed with a mean value 0 and standard deviation σ;
- the polar angles are random, independent and uniformly distributed in $[-\pi; +\pi]$;
- the standard deviation of modern CMM is $\sigma < 10^{-4}$ [14].

Let r_i, θ_i and ϕ_i be the deviations of the i^{th} feature coordinates. As they are random variables the position tolerance deviations will also be random variables with standard deviations σ_{L_i}. Our goal is to find σ_{L_i} using the value of radius standard uncertainty σ.

The implementation of Monte-Carlo simulation resides on two probability theorems:

Theorem 1 (Linearity of Expectation). *[15, p. 8]: Let $Ex(R_1)$ and $Ex(R_2)$ be the expected values of random variables R_1 and R_2. Then for every pair of R_1 and R_2 the expected value of their sum is:*

$$Ex(R_1 + R_2) = Ex(R_1) + Ex(R_2).$$

Theorem 2 (Function Linearization). *: Let n be the set of uncorrelated random variables $\{X_1, \ldots, X_n\}$ with expectations m_i and dispersions σ_i^2. Let $Z = H(X_1, \ldots, X_n)$. Then expectation and dispersion of Z are approximately equal to:*

$$m_Z \approx H(m_1, \ldots, m_n) + \frac{1}{2} \sum_{i=1}^{n} \frac{\partial^2 H}{\partial x_i^2} \sigma_i^2,$$

$$\sigma_Z^2 \approx \sum_{i=1}^{n} \left(\frac{\partial H}{\partial x_i}\right)^2 \sigma_i^2,$$

where all partial derivatives of H are evaluated at $(m_1, \ldots, m_n)$.

Function $H(X_1, \ldots, X_n)$ is defined by the position tolerance fitting criteria. The position tolerance is about finding relationship between two coordinate systems using specific criterion. In this paper a minimization of the sum of the squares is used:

$$L_{\mathrm{LSQ}} = H(r_{a_1}, \ldots, r_{a_N}, r_{n_1}, \ldots, r_{n_N}) =$$

$$= \min_{\substack{\alpha \in [-\pi; +\pi] \\ t \in \mathbb{R}^2}} \sum_{i=1}^{N} (r_{n_i} - T(r_{a_i}; \alpha; t))^2, \tag{1}$$

where $T(r_{a_i}; \alpha; t)$ – planar absolute orientation transformation; r_{a_i} – measured coordinates vector; r_{n_i} – nominal coordinates vector; α – rotation angle; t – translation vector.

Planar absolute orientation transformation is evaluated using orthonormal matrix [17]:

$$T(r_a; \alpha; t) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} r_a + t.$$

[2]Averaging fit is not in accordance with the definitions given in GD&T standards and does not ensure assemblability.

Table 1: Coordinates for the symmetric pattern

Point No	X, mm	Y, mm	Point No	X, mm	Y, mm
1	55.20	31.88	10	33.12	-57.38
2	33.12	57.38	11	55.20	-31.88
3	0.0	63.75	12	33.12	57.38
4	-33.12	57.38	13	66.24	0.0
5	-55.20	31.88	14	33.12	6.38
6	-66.24	0.0	15	11.04	31.88
7	-55.20	-31.88	16	-22.08	25.50
8	-33.12	-57.38	17	-33.12	-6.38
9	0.0	-63.75	18	-11.04	-31.88
			19	22.08	-25.50

As shown by Horn [16] minimization of the equation (1) can be substituted by the minimization of

$$L_{\text{LSQ}} = \min_{\alpha\in[-\pi;+\pi]} \sum_{i=1}^{N} \left(r'_{n_i} - T\left(r'_{a_i};\alpha;0\right)\right)^2, \tag{2}$$

in the coordinate system where centroids of nominal and measured datasets are equal. A closed-form solution proposed by Horn et al. [17] was not used in this paper since most of CMM software implementations rely on a generic minimization process rather than on ad hoc methods.

Jacobian for Levenberg—Marquardt [21] minimization of (2) was computed using finite differences:

$$J_F = \left(\frac{D(\alpha+\varepsilon) - D(\alpha)}{\varepsilon} \right),$$

where $D(\alpha) = |r'_{n_i} - T(r'_{a_i};\alpha;0)|$ and ε was chosen to be 10^{-11} which provided numerical stability of the results.

5. Implementation details

Minimization of the sum of the squares was done using Levenberg—Marquardt algorithm implementation from ALGLIB [24] optimization library (based on Minpack [25]).

To evaluate a position tolerance deviations one needs nominal points coordinates and measured points coordinates, usualy from a CMM. CMM uncertainty is composed by many factors that can be put in several major groups [10]: influence of CMM hardware, workpiece form deviation, distribution of the measured points, evaluation algorithm and its implementation. To generate uniformly distributed values of θ_i and ϕ_i a linear congruential generator [22] is used. To generate normally distributed values of r_i Law—Kelton polar transform [23] was applied as proposed by Dhanish and Mathew [10] in their study.

The number of Monte-Carlo trials for every simulated experiment was chosen according to [4, Sub Clause 7.2.1] and with confidence level 0.95 equals $M = 10^6$ trials.

Table 2: Coordinates for the asymmetric long pattern

Point No	X, mm	Y, mm
1	0.0	0.0
2	584.4	-834.6
3	584.4	-1911.6
4	0.0	-2746.2

Let L_i be the result of i^{th} Monte-Carlo trial of equation (1); then according to [4, Sub Clause 7.6], the mean value:

$$\tilde{L} = \frac{1}{M}\sum_{i=1}^{M} L_i \tag{3}$$

is used as the measurement result, and the value:

$$u^2(\tilde{L}) = \frac{1}{M-1}\sum_{i=1}^{M} \left(L_i - \tilde{L}\right)^2 \tag{4}$$

as the standard uncertainty $u(\tilde{L})$ of this result.

The extended uncertainty was calculated using 2.5% and 97.5% percentiles for a confidence level of 0.95. The difference of the percentiles is used as the width of the uncertainty inverval.

The standard uncertainties of the input data are (U_{in}, μm): 1.7; $1.7 + \frac{L}{350}$; $1.7 + \frac{L}{200}$; 2.4; $2.4 + \frac{L}{350}$; $2.4 + \frac{L}{200}$; where L is the distance from the origin to the corrent bore in the pattern, *mm*.

Simulation software was written in Free Pascal. Simulation is performed on an Intel Core i7 2.6 GHz machine. The runtime for each value of input uncertainty with 10^6 trials is within 3 minutes.

6. Evaluated data sets

Since no reference data sets are available for position tolerance evaluation hence drawings and numerical values of such data sets are proposed.

Simulation was made using three different data sets, where point coordinates correspond to centers of bores in a workpiece:

1. Symmetric pattern (Star) — coordinates of 19 symmetricaly arranged bores (table 1).
2. Asymmetric long pattern — coordinates of 4 assymetricaly arranged bores on long distance (table 2).
3. Asymmetric short pattern (rectangular) — coordinates of 6 bores in rectanguălar grid (table 3).

The proposed bores configurations cover different types of workpieces typically involved in position tolerance evaluation.

Table 3: Coordinates for the rectangular pattern

Point No	X, mm	Y, mm
1	-10.0	25.0
2	-50.0	25.0
3	-10.0	65.0
4	-50.0	65.0
5	-10.0	105.0
6	-50.0	105.0

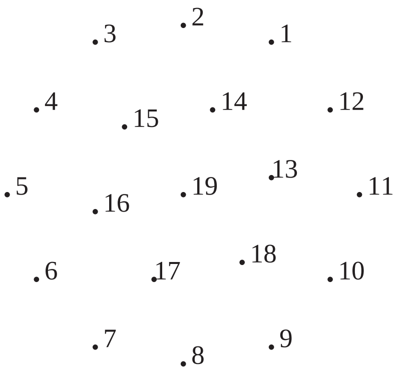

Fig. 1: Drawing of the symmetric pattern

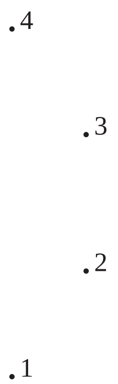

Fig. 2: Drawing of the asymmetric long pattern

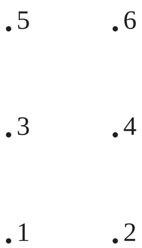

Fig. 3: Drawing of the rectangular pattern

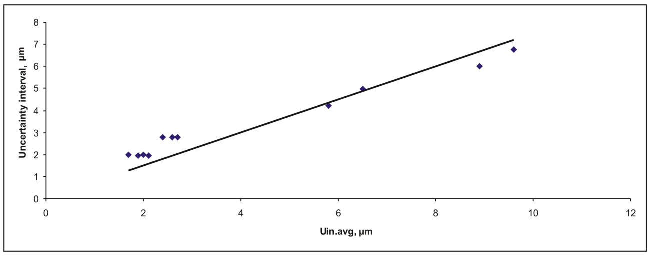

Fig. 4: Function $U_{95}(U_{in.avg})$ for different patterns.

7. Results

The histogram of series of Monte-Carlo trials is a plot of the measured position tolerance deviation versus the number of measurements at that deviation. The histogram can be used to model the probability distribution of the measurements. In the evaluated model, if the number of trials is increased, then the histogram begins to take on some definite shape. As the number of trials approaches infinity, the histogram distribution approaches some definite, continuous curve, called the limiting distribution [26]. Different types of models result in different limiting distributions.

Histograms of position tolerance deviations for each data set are presented in figures 5—7.

The dependancy $U_{95}(U_{in.avg})$ for different patterns is presented in figure 4. Horizontal axis corresponds to average input uncertainty, vertical axis — simulated width of uncertainty interval.

Final results of Monte-Carlo simulation for different values of input uncertainty are in table 4.

8. Conclusion

Using the wide spread theoretical position tolerance model and based on Gaussian distribution the input CMM uncertainties are propagated to the position tolerance uncertainty. The dependencies constructed show that uncertainty transformation is linear and significant uncertainty values arise for long workpieces of several meters length.

9. Acknowledgements

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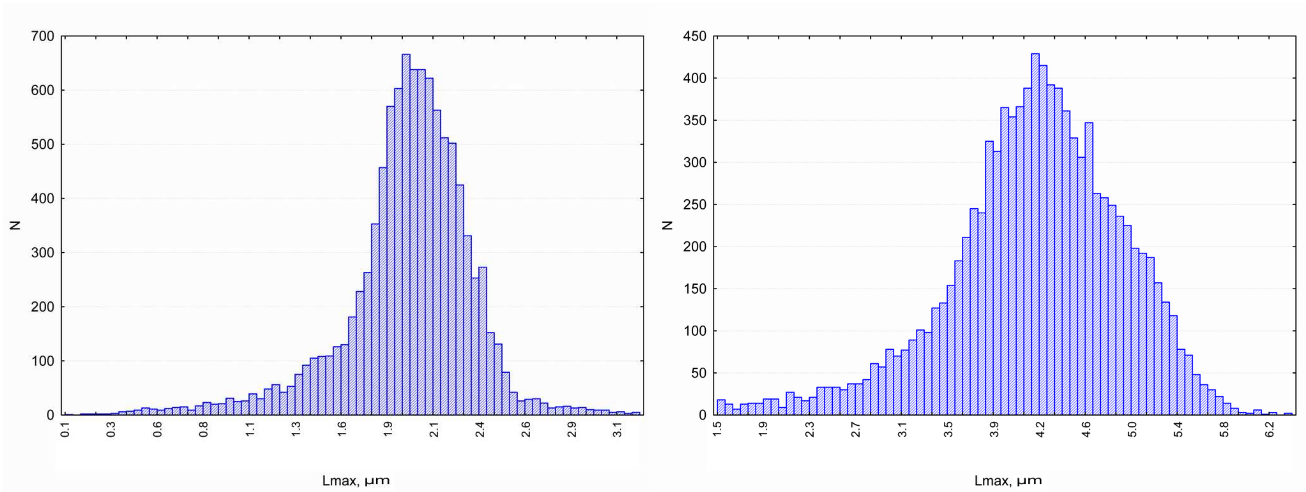

Fig. 5: Asymmetric long pattern, (a) $U_{in} = 1.7\ \mu m$, (b) $U_{in} = 1.7 + \frac{L}{350}\ \mu m$.

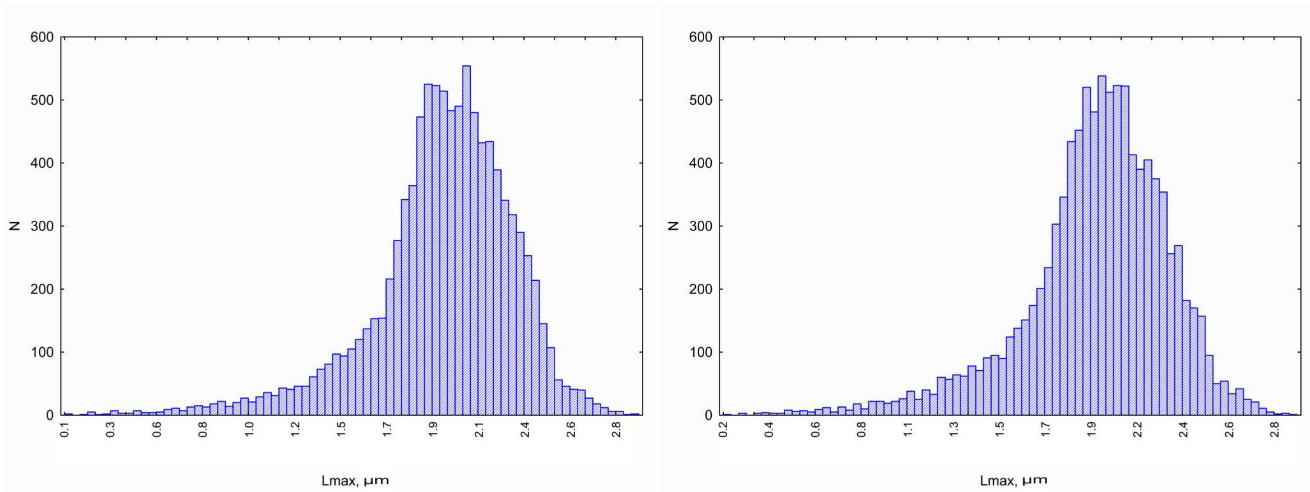

Fig. 6: Rectangular pattern, (a) $U_{in} = 1.7\ \mu m$, (b) $U_{in} = 1.7 + \frac{L}{350}\ \mu m$.

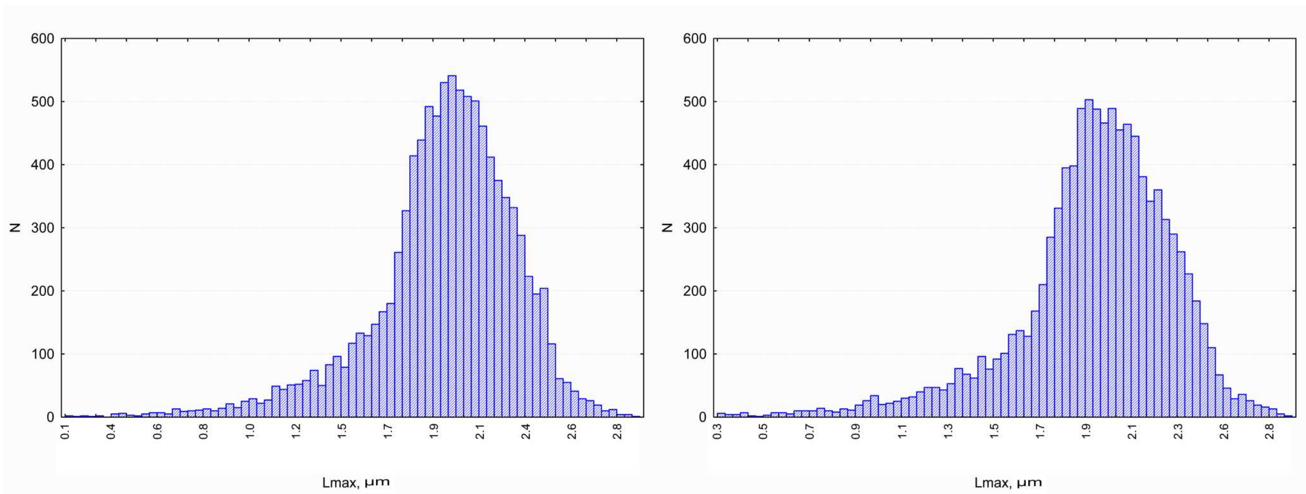

Fig. 7: Symmetric pattern, (a) $U_{in} = 1.7\ \mu m$, (b) $U_{in} = 1.7 + \frac{L}{350}\ \mu m$.

Table 4: Simulation results

U_{in}, μm	$U_{in.avg}$, μm	Mean, μm	Std.deviation, μm	Skewness	Kurtosis	Mode, μm	Uncertainty interval U_{95}, μm
Symmetric pattern							
1.7	1.7	1.98	0.35	-1.05	2.26	2.00	1.80
$1.7 + \frac{L}{350}$	1.9	1.97	0.36	-1.11	2.32	2.11	1.47
$2.4 + \frac{L}{350}$	2.6	2.77	0.48	-1.27	2.52	2.87	1.95
Rectangular pattern							
1.7	1.7	1.97	0.35	-1.09	2.43	1.93	1.45
$1.7 + \frac{L}{350}$	1.9	1.97	0.35	-1.01	2.02	2.01	1.45
$2.4 + \frac{L}{350}$	2.6	2.76	0.49	-1.28	2.57	2.87	1.98
Assymetric long pattern							
1.7	1.7	1.97	0.35	-0.99	3.03	2.00	1.48
$1.7 + \frac{L}{350}$	5.8	4.22	0.75	-0.61	0.78	4.29	3.10
$2.4 + \frac{L}{350}$	6.5	4.99	0.89	-0.75	1.33	4.95	3.69

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