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Measuring the Moment and the Magnitude of the Abrupt Change of the Gaussian Process Bandwidth

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The maximum likelihood algorithm is introduced for measuring the unknown moment of abrupt change and bandwidth jump of a fast-fluctuating Gaussian random process. This algorithm can be technically implemented much simpler than the ones obtained by means of common approaches. The technique for calculating the characteristics of the synthesized measurer is presented and the closed analytical expressions for the conditional biases and variances of the resulting estimates are found using the additive local Markov approximation of the decision statistics. By statistical simulation methods, it is confirmed that the presented measurer is operable, while the theoretical formulas describing its performance well approximate the corresponding experimental data in a wide range of the parameter values of the analyzed random process.

Keywords: Abrupt change of random process, unknown bandwidth, maximum likelihood method, discontinuous parameter, local Markov approximation method, bias of estimate, variance of estimate, statistical simulation.

1. INTRODUCTION

The problem of the statistical analysis of the abrupt change (i.e., instantaneous jumping) of the parameter values of a random process at some moment in time is studied in a number of papers [1]-[5]. In certain publications, the statement of this problem is accompanied by the assumption that the observable data realization has a normal distribution. As a rule, the additional restrictions are also imposed referring to the processed samples being uncorrelated (and therefore independent) [1], [2] or to the specified model classes of the information signal [2]-[5], etc.

In this paper, we propose a technically simple method for measuring the unknown moment of abrupt change and bandwidth jump of a Gaussian random process when only two conditions for the analyzed process are satisfied: that its fluctuations are fast and that its spectral density is approximately uniform within the specified bandwidth. The characteristics of the synthesized measurer are found both theoretically and experimentally.

2. THE PROBLEM STATEMENT

Let us define analytically the band fast-fluctuating Gaussian random process with the bandwidth jump at the moment in time λ_0 as follows

$$\xi(t)=[1-\theta(t-\lambda_0)]\nu_1(t)+\theta(t-\lambda_0)\nu_2(t),\ \theta(t)=\begin{cases}1,\ t\ge 0,\\ 0,\ t<0,\end{cases} \quad (1)$$

where $\nu_i(t)$, $i=1,2$ – statistically independent stationary Gaussian random processes with the mathematical expectations a and the spectral densities [6], [7]

$$G_i(\omega)=\frac{d}{2}\begin{cases}1,\ |\omega|\le \Omega_{0i}/2,\\ 0,\ |\omega|>\Omega_{0i}/2.\end{cases}$$

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Here Ω_{0i} is the bandwidth, and d is the intensity (spectral density magnitude) of the process $\nu_i(t)$ determining its dispersion $D_i = d\Omega_{0i}/4\pi$, while $\Omega_{01} \neq \Omega_{02}$. This type of spectral density shape approximation can be used if the conditions $\Delta\Omega_i << (2\pi/\tau_i) << \Omega_{0i}$ are satisfied [6]. In the latter formula $\Delta\Omega_i$ is the bandwidth within which the real spectral density $G_i(\omega)$ decreases from its maximum value to almost zero, $\tau_1 = \lambda_0$, $\tau_2 = T - \lambda_0$ and T is the observation interval of the random process $\xi(t)$ (1).

We presuppose that the process (1) is observed against Gaussian white noise $n(t)$ with one-sided spectral density N_0, so that the additive mix

$$x(t) = \xi(t) + n(t), \qquad t \in [0, T] \tag{2}$$

arrives at the input of the receiver.

The parameters λ_0 and Ω_{02} are unknown and possess the values from the prior intervals $[\Lambda_1, \Lambda_2]$, $[\Upsilon_1, \Upsilon_2]$. And we consider that $\Upsilon_1 < \Omega_{01} < \Upsilon_2$, while the condition of the "fast" fluctuations of the process $\xi(t)$ is stated as

$$\mu_{\min} = T_{\min} \Upsilon_1 / 4\pi >> 1, \tag{3}$$

where $T_{\min} = \min(\lambda_0, T - \lambda_0)$.

With the observed realization (2) and the available prior information, it is necessary to estimate the moment of abrupt change λ_0 and the bandwidth Ω_{02} of the process $\xi(t)$ after the abrupt change.

3. The synthesis of the estimation algorithm

In order to synthesize the algorithm for estimating the moment and the magnitude of the abrupt change of the process $\xi(t)$ (1) bandwidth we apply the maximum likelihood method. According to this method, it is necessary to form the decision statistics – the logarithm of the functional of the likelihood ratio (FLR) – as a function of the current values of all the unknown parameters. If the inequality (3) holds, then according to [6]-[8] we have

$$L(\lambda, \Omega) = \frac{d}{N_0 (N_0 + d)} \left[\int_0^{\lambda} y^2(t, \Omega_{01}) \, dt + \int_{\lambda}^{T} y^2(t, \Omega) \, dt \right] + \frac{2a}{N_0 + d} \int_0^T x(t) \, dt - \frac{a^2 T}{N_0 + d} - \frac{(T - \lambda)\Omega + \lambda \Omega_{01}}{4\pi} \ln\left(1 + \frac{d}{N_0}\right). \tag{4}$$

Here $y(t, \Theta) = \int_{-\infty}^{\infty} x(t') h(t - t', \Theta) dt'$ is the output signal of the filter with the transfer function $H(\omega, \Theta)$ satisfying the condition $|H(\omega, \Theta)|^2 = 1$, if $|\omega| \leq \Theta/2$, and $|H(\omega, \Theta)|^2 = 0$, if $|\omega| > \Theta/2$, while λ, Ω are the current values of the unknown parameters λ_0, Ω_{02}, respectively.

The maximum likelihood estimates (MLEs) λ_m, Ω_m of the measured values λ_0, Ω_{02} are determined as the position of the greatest maximum of the decision statistics (4):

$$(\lambda_m, \Omega_m) = \underset{\lambda \in [\Lambda_1, \Lambda_2], \Omega \in [\Upsilon_1, \Upsilon_2]}{\arg\max} L(\lambda, \Omega). \tag{5}$$

In practice, the maximum likelihood measurer (5) can be implemented as an N-channel device, each channel of which is matched to the bandwidth $\Omega_i = \Upsilon_1 + (i - 1/2)\Delta\Omega$, $i = \overline{1, N}$, $\Delta\Omega = (\Upsilon_2 - \Upsilon_1)/N$. The block diagram of such a device is shown in Fig. 1. Here the designations are: 1 is the switch that is open for time $[0, T]$; 2^0 is a filter with transfer function $H(\omega, \Omega_{01})$; 2^i is a filter with transfer function $H(\omega, \Omega_i)$; 3 is the squarer; 4 is the subtractor; 5 is an integrator over the time interval $[0, T]$; 6 is the delay line for time T; 7 is an integrator; 8 is the ramp generator; 9 is the multiplier; 10 is the resolver that determines both the estimate of the bandwidth after the abrupt change by the channel number with the maximum response magnitude and the estimate of the moment of abrupt change by the position of the greatest maximum of the signal in this channel within the interval $[\Lambda_1, \Lambda_2]$. It is obvious that the greater the number of channels N, the more accurate the measurer presented in Fig.1. implements the algorithm (5).

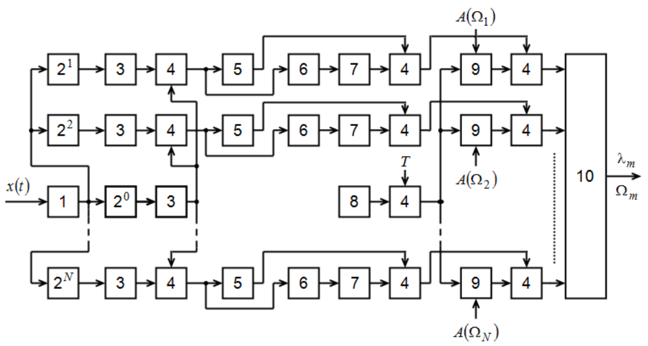

Fig.1. The maximum likelihood measurer of the moment and the magnitude of the abrupt change in the bandwidth of a Gaussian random process.

4. The characteristics of the estimation algorithm

Let us find the characteristics of the measurer (5). For this purpose, we move from MLEs λ_m, Ω_m (5) to normalized MLEs $l_m = 1 - \lambda_m/T$, $v_m = \Omega_m/\Omega_{01} - 1$ defined as

$$(l_m, v_m) = \underset{l \in [1 - \tilde{\Lambda}_2, 1 - \tilde{\Lambda}_1], v \in [\tilde{\Upsilon}_1 - 1, \tilde{\Upsilon}_2 - 1]}{\arg\max} M(l, v). \tag{6}$$

Here

$$M(l, v) = M_1(l, v) - M_2(l) - S_3(l, v), \tag{7}$$

$$M_1(l,v)=\frac{T}{\mu_1 N_0}\int_{1-l}^{1} y^2\left(T\tilde{t},\Omega_{01}(v+1)\right)\mathrm{d}\tilde{t}\ ,$$
$$M_2(l)=\frac{T}{\mu_1 N_0}\int_{1-l}^{1} y^2\left(T\tilde{t},\Omega_{01}\right)\mathrm{d}\tilde{t}\ , \qquad (8)$$
$$S_3(l,v)=lv(1+q)\ln(1+q)/q\ ,$$

and

$$\tilde{t}=t/T\ ,\qquad v=\Omega/\Omega_{01}-1\ ,\qquad q=d/N_0\ , \qquad (9)$$

$$l=1-\lambda/T\ ,\ \tilde{\Lambda}_{1,2}=\Lambda_{1,2}/T\ ,\ \tilde{\Upsilon}_{1,2}=\Upsilon_{1,2}/\Omega_{01}\ ,\ \mu_1=T\Omega_{01}/4\pi\ .$$

If the condition (3) is satisfied, then the functionals $M_1(l,v)$, $M_2(l)$ (8), and therefore the functional $M(l,v)$ (7) are Gaussian ones approximately [6]. Thus, they can be completely described in the statistical sense by means of the moment or correlation functions of the first two orders. According to this, we present them as the sum of regular and fluctuation components [9], [10]:

$$M_1(l,v)=S_1(l,v)+N_1(l,v),\quad M_2(l)=S_2(l)+N_2(l).$$

Here $S_1(l,v)=\langle M_1(l,v)\rangle$, $S_2(l)=\langle M_2(l)\rangle$ are regular, $N_1(l,v)=M_1(l,v)-\langle M_1(l,v)\rangle$, $N_2(l)=M_2(l)-\langle M_2(l)\rangle$ are fluctuation components, and the averaging $\langle\cdot\rangle$ is performed in terms of all possible realizations $x(t)$ with fixed values for λ_0, Ω_{02}. By directly averaging (8), we find

$$S_1(l,v)=l(v+1)+q\max(0,l-l_0)\left[1+\min(0,v)\right]+ \\ +q\min(l_0,l)\left[1+\min(v_{02},v)\right],$$
$$S_2(l)=l+q\max(0,l-l_0)+ \\ +q\min(l_0,l)\left[1+\min(0,v_{02})\right],$$
$$\langle N_1(l_1,v_1)N_1(l_2,v_2)\rangle=\{\min(l_1,l_2)(1+\min(v_1,v_2))+ \\ +q(2+q)\left[\max(0,\min(l_1,l_2)-l_0)(1+\min(0,v_1,v_2))+ \\ +\min(l_0,l_1,l_2)(1+\min(v_{02},v_1,v_2))\right]\}/\mu_1\ , \qquad (10)$$
$$\langle N_2(l_1)N_2(l_2)\rangle=\{\min(l_1,l_2)+q(2+q)\times \\ \times\left[\max(0,\min(l_1,l_2)-l_0)+ \\ (1+\min(0,v_{02}))\min(l_0,l_1,l_2)\right]\}/\mu_1\ ,$$

where $l_0=1-\lambda_0/T$, $v_{02}=\Omega_{02}/\Omega_{01}-1$.

Using (10), we write the regular component $S(l,v)=\langle M(l,v)\rangle$ and the correlation function of the fluctuation component $N(l,v)=M(l,v)-\langle M(l,v)\rangle$ of the decision statistics $M(l,v)$ (7) as follows:

$$S(l,v)=\left[1-(1+q)\ln(1+q)/q\right]lv-q\min(0,l_0-l)\times \\ \times\min(0,v)+q\min(l_0,l)\left[\min(v_{02},v)-\min(0,v_{02})\right],$$
$$\langle N(l_1,v_1)N(l_2,v_2)\rangle=(1/\mu_1)\{\min(l_1,l_2)\left[\min(v_1,v_2)-\right. \\ -\min(0,v_1)-\min(0,v_2)\left.\right]+q(2+q)\min(l_0,l_1,l_2)\times \\ \times\left[\min(v_{02},v_1,v_2)+\min(0,v_{02})-\min(0,v_{02},v_1)-\right. \\ -\min(0,v_{02},v_2)\left.\right]+q(2+q)\max(0,\min(l_1,l_2)-l_0)\times \\ \times\left[\min(0,v_1,v_2)-\min(0,v_1)-\min(0,v_2)\right]\}. \qquad (11)$$

We take into account that the regular component $S(l,v)$ reaches the absolute maximum at the point (l_0,v_{02}), while the realizations of the fluctuation component $N(l,v)$ are continuous with probability 1. Then the output signal-to-noise ratio (SNR) for the algorithm (5), (6) is determined as [7], [9]

$$z^2=\frac{S^2(l_0,v_0)}{\langle N^2(l_0,v_0)\rangle}=\mu_1 l_0|v_{02}|\left[1-\frac{1+c_q}{q}\ln(1+q)\right]^2, \qquad (12)$$

where $c_q=0$, if $v_{02}>0$, and $c_q=q$, if $v_{02}<0$.

From (12) it follows that the SNR $z^2>>1$, if the inequality (3) is satisfied and the value of q is not too small. In this case, the coordinates (l_m,v_m) (6) of the position of the absolute maximum of the functional $M(l,v)$ (7) are situated in a near δ-neighborhood of the point (l_0,v_{02}). While increasing z^2 ($z^2\to\infty$), the size of this neighborhood $\delta=\max(|l-l_0|,|v-v_{02}|)\to 0$ [6], [9], and for the regular component and the correlation function of the fluctuation component (11), the asymptotic representations are valid:

$$S(l,v)=S_0+S_1(l-l_0)+S_2(v-v_{02})+o(\delta),$$
$$\langle N(l_1,v_1)N(l_2,v_2)\rangle=\sigma^2+R_1(l_1-l_0,l_2-l_0)+ \\ +R_2(v_1-v_{02},v_2-v_{02})+o(\delta), \qquad (13)$$

where

$$S_0=l_0 v_{02}\begin{cases}(1+q)\left[1-\ln(1+q)/q\right], & v_{02}>0,\\ 1-(1+q)\ln(1+q)/q, & v_{02}<0,\end{cases}$$
$$\sigma^2=\frac{l_0|v_{02}|}{\mu_1}\begin{cases}(1+q)^2, & v_{02}>0,\\ 1, & v_{02}<0,\end{cases} \qquad (14)$$

$$S_1(x)=v_{02}\begin{cases}x\left[1-(1+q)\ln(1+q)/q\right]+q\min(0,x), & v_{02}>0,\\ x\left[1-(1+q)\ln(1+q)/q\right]+q\max(0,x), & v_{02}<0,\end{cases}$$
$$S_2(y)=l_0\left[y(1-(1+q)\ln(1+q)/q)+q\min(0,y)\right],$$

$$R_1(x_1,x_2)=\frac{|v_{02}|}{\mu_1}\begin{cases}\min(x_1,x_2)+q(2+q)\min(0,x_1,x_2), & v_{02}>0,\\ \min(x_1,x_2)+q(2+q)\max(0,\min(x_1,x_2)), & v_{02}<0,\end{cases} \quad (15)$$

$$R_2(y_1,y_2)=\frac{l_0}{\mu_1}\begin{cases}q(2+q)\min(0,y_1,y_2)+\min(y_1,y_2), & v_{02}>0,\\ q(2+q)\left[\min(0,y_1,y_2)-\min(0,y_1)-\right. \\ \left.-\min(0,y_2)\right]-\max(y_1,y_2), & v_{02}<0.\end{cases}$$

We introduce the statistically independent Gaussian random processes $r_1(l)$, $r_2(v)$ with mathematical expectations $\langle r_1(l)\rangle = S_1(l-l_0)$, $\langle r_2(v)\rangle = S_2(v-v_{02})$ and correlation functions

$$\left\langle [r_1(l_1)-\langle r_1(l_1)\rangle][r_1(l_2)-\langle r_1(l_2)\rangle]\right\rangle = \sigma^2/2 + R_1(l_1-l_0, l_2-l_0),$$

$$\left\langle [r_2(v_1)-\langle r_2(v_1)\rangle][r_2(v_2)-\langle r_2(v_2)\rangle]\right\rangle = \sigma^2/2 + R_2(v_1-v_{02}, v_2-v_{02}).$$

If $z\to\infty$ (12) and $\delta\to 0$, then the asymptotically Gaussian random field $[M(l,v)-S_0]$ converges in distribution to the sum $r_1(l)+r_2(v)$. Therefore, while increasing SNR z (12) the normalized estimates l_m, v_m converge in distribution to the corresponding estimates

$$\eta_{m1} = \underset{l\in[l_0-\delta,l_0+\delta]}{\arg\max}\, r_1(l), \qquad \eta_{m2} = \underset{v\in[v_{02}-\delta,v_{02}+\delta]}{\arg\max}\, r_2(v),$$

and MLE λ_m and Ω_m (5) converge in distribution to the random variables

$$T(1-\eta_{m1}) \text{ and } \Omega_{01}(\eta_{m2}+1). \quad (16)$$

From (13)-(15) it follows that within the intervals $[l_0-\delta, l_0+\delta]$, $[v_{02}-\delta, v_{02}+\delta]$ conditions of the Doob's theorem [11] are satisfied for the processes $r_1(l)$, $r_2(v)$, so they are continuous Gaussian Markov processes with drift coefficients $a_{\eta1}$, $a_{\eta2}$ and diffusion coefficients $b_{\eta1}$, $b_{\eta2}$:

$$a_{\eta1}=|v_{02}|\begin{cases}S_{11}, & l<l_0,\\ -S_{12}, & l\ge l_0,\end{cases} \qquad a_{\eta2}=l_0\begin{cases}S_{21}, & v<v_{02},\\ -S_{22}, & v\ge v_{02},\end{cases}$$

$$b_{\eta1}=\begin{cases}\sigma_{11}^2, & l<l_0,\\ \sigma_{12}^2, & l\ge l_0,\end{cases} \qquad b_{\eta2}=\begin{cases}\sigma_{21}^2, & v<v_{02},\\ \sigma_{22}^2, & v\ge v_{02}.\end{cases}$$

Here

$$S_{11}=\begin{cases}S_{21}, & v_{02}>0,\\ S_{22}, & v_{02}<0,\end{cases} \qquad S_{12}=\begin{cases}S_{22}, & v_{02}>0,\\ S_{21}, & v_{02}<0,\end{cases}$$

$$S_{21}=(1+q)[1-\ln(1+q)/q], \quad S_{22}=(1+q)\ln(1+q)/q-1,$$

$$\sigma_{11}^2=\frac{|v_{02}|}{\mu_1}\begin{cases}(1+q)^2, & v_{02}>0,\\ 1, & v_{02}<0,\end{cases} \quad \sigma_{12}^2=\frac{|v_{02}|}{\mu_1}\begin{cases}1, & v_{02}>0,\\ (1+q)^2, & v_{02}<0,\end{cases}$$

$$\sigma_{21}^2 = l_0(1+q)^2/\mu_1, \qquad \sigma_{22}^2 = l_0/\mu_1.$$

In [12], the analytical expressions have been found for the statistical characteristics of the magnitude and the position of the greatest maximum of Markov random process with piecewise constant drift and diffusion coefficients. Referring to the results of the studies [12], for the probability densities $w_{\eta i}(\eta)$ of the random variables η_{mi}, $i=1,2$ we obtain

$$w_i(\eta)=\begin{cases}z_{i1}^2\Psi_g\left(z_{i1}^2(\eta_{0i}-\eta), z_{i1}^2\delta, z_{i2}^2\delta, 1/R_i\right), & \eta<\eta_{0i},\\ z_{i2}^2\Psi_g\left(z_{i2}^2(\eta-\eta_{0i}), z_{i2}^2\delta, z_{i1}^2\delta, R_i\right), & \eta\ge\eta_{0i},\end{cases} \quad (17)$$

where

$$z_{1i}^2 = 2v_{02}^2 S_{1i}^2/\sigma_{1i}^2, \quad z_{2i}^2 = 2l_0^2 S_{2i}^2/\sigma_{2i}^2, \quad \eta_{01}=l_0, \quad (18)$$

$$\eta_{02}=v_{02}, \quad R_1 = S_{11}\sigma_{12}^2/S_{12}\sigma_{11}^2, \quad R_2 = S_{21}\sigma_{22}^2/S_{22}\sigma_{21}^2,$$

$$\Psi_g(y,y_1,y_2,y_3)=\frac{1}{2\sqrt{\pi}|y|^{3/2}}\left\{\frac{\exp[-(y_1-y)/4]}{\sqrt{\pi(y_1-y)}}+\Phi\left(\sqrt{\frac{y_1-y}{2}}\right)\right\}\times$$
$$\times\int_0^\infty \xi\exp\left[-\frac{(\xi+y)^2}{4y}\right]\left[\Phi\left(\frac{y_3\xi+y_2}{\sqrt{2y_2}}\right)-\exp(-y_3\xi)\Phi\left(\frac{-y_3\xi+y_2}{\sqrt{2y_2}}\right)\right]d\xi.$$

The expression (17) is not suitable for practical calculations due to the difficulty of determining the value of δ. In this regard, we note that according to (18) $z_{1i}^2\to\infty$, $z_{2i}^2\to\infty$, if $\mu_1\to\infty$ and $q>0$. Thus, if the condition (3) holds and the value of q is not too small, then we can consider $z_{1i}^2 >> 1$, $z_{2i}^2 >> 1$ and, similarly to [12], instead of (17), use a simpler approximation of the form of

$$w_i(\eta)=\begin{cases}z_{i1}^2\Psi\left(z_{i1}^2(\eta_{0i}-\eta), 1/R_i\right), & \eta<\eta_{0i},\\ z_{i2}^2\Psi\left(z_{i2}^2(\eta-\eta_{0i}), R_i\right), & \eta\ge\eta_{0i},\end{cases} \quad (19)$$

where

$$\Psi(x,y)=\Phi\left(\sqrt{|x|/2}\right)-1+(2y+1)\exp\left[|x|y(y+1)\right]\times$$
$$\times\left[1-\Phi\left((2y+1)\sqrt{|x|/2}\right)\right].$$

The accuracy of the formula (19) increases with $\mu_{\min}$ (3) and z_{1i}, z_{2i} (18).

Using (16), (19) it is easy to write asymptotic expressions for the conditional biases $b(\lambda_m|\lambda_0)=\langle\lambda_m-\lambda_0\rangle$, $b(\Omega_m|\Omega_{02})=\langle\Omega_m-\Omega_{02}\rangle$ and variances

$V(\lambda_m|\lambda_0) = \langle(\lambda_m - \lambda_0)^2\rangle$, $V(\Omega_m|\Omega_{02}) = \langle(\Omega_m - \Omega_{02})^2\rangle$ of the estimates (5):

$$
\begin{aligned}
b(\lambda_m|\lambda_0) &= -T\langle\eta_{m1} - \eta_{01}\rangle, \\
b(\Omega_m|\Omega_{02}) &= \Omega_{01}\langle\eta_{m2} - \eta_{02}\rangle, \\
V(\lambda_m|\lambda_0) &= T^2\langle(\eta_{m1} - \eta_{01})^2\rangle, \\
V(\Omega_m|\Omega_{02}) &= \Omega_{01}^2\langle(\eta_{m2} - \eta_{02})^2\rangle,
\end{aligned} \tag{20}
$$

where

$$
\begin{aligned}
\langle\eta_{m1} - \eta_{01}\rangle &= \int_{1-\tilde{\Lambda}_2}^{1-\tilde{\Lambda}_1} (\eta - \eta_{01}) w_1(\eta) d\eta, \\
\langle(\eta_{m1} - \eta_{01})^2\rangle &= \int_{1-\tilde{\Lambda}_2}^{1-\tilde{\Lambda}_1} (\eta - \eta_{01})^2 w_1(\eta) d\eta, \\
\langle\eta_{m2} - \eta_{02}\rangle &= \int_{\tilde{\Upsilon}_1 - 1}^{\tilde{\Upsilon}_2 - 1} (\eta - \eta_{02}) w_2(\eta) d\eta, \\
\langle(\eta_{m2} - \eta_{02})^2\rangle &= \int_{\tilde{\Upsilon}_1 - 1}^{\tilde{\Upsilon}_2 - 1} (\eta - \eta_{02})^2 w_2(\eta) d\eta.
\end{aligned} \tag{21}
$$

The exact values of the integrals (21) with fixed z_{1i}, z_{2i} can only be found using numerical computing. However, if the conditions $z_{1i} >> 1$, $z_{2i} >> 1$ are satisfied, then, following [12], we can propose the simpler asymptotic approximations for the biases and the variances of the estimates (5) instead of (20), (21). Indeed, in this case, the function $w_i(\eta)$ is significantly different from zero in a small neighborhood of the point η_{0i} so that, without significant accuracy losses, the limits of integration in (21) can be extended to infinity. Then, after performing the corresponding mathematical operations we get

$$
\langle\eta_{mi} - \eta_{0i}\rangle = \frac{z_{1i}^2 R_i(R_i + 2) - z_{2i}^2(2R_i + 1)}{z_{1i}^2 z_{2i}^2 (R_i + 1)^2}, \tag{22}
$$

$$
\langle(\eta_{mi} - \eta_{0i})^2\rangle = \frac{2\left[z_{1i}^4 R_i(2R_i^2 + 6R_i + 5) + z_{2i}^4(5R_i^2 + 6R_i + 2)\right]}{z_{1i}^4 z_{2i}^4 (R_i + 1)^3}.
$$

The accuracy of the formulas (22) increases with $\mu_{\min}$ (3) and z_{1i}, z_{2i} (18). For small values of z_{1i}, z_{2i}, the calculations carried out by means of (22) can lead to large errors as the formulas (22), in contrast to (21), do not take into account the finite length of the prior intervals of the possible values of the unknown parameters λ_m, Ω_m.

5. Results of the statistical simulation

In order to establish the borders of applicability for the found approximate formulas for the characteristics of the synthesized maximum likelihood estimation algorithm, we demonstrate the statistical computer simulation of the measurer (5). During simulation within the interval $\tilde{t} \in [1 - \tilde{\Lambda}_2, 1]$ (9), the samples $\tilde{y}_{kp} = y(T\tilde{t}_k, \Omega_{01}(v_p + 1))\sqrt{T/N_0}$ (8) are formed at discrete points in time $\tilde{t}_k = 1 - \tilde{\Lambda}_2 + k\Delta\tilde{t}$, $k = \overline{0, \mathrm{int}\{\tilde{\Lambda}_2/\Delta\tilde{t}\}}$ for every significant value of $v_p = \tilde{\Upsilon}_1 - 1 + p\Delta v$, $p = \overline{0, \mathrm{int}\{(\tilde{\Upsilon}_2 - \tilde{\Upsilon}_1)/\Delta v\}}$ of the normalized bandwidth v (9), as it is described in [10]. In terms of the formed samples $\tilde{y}_{kp}$, following [10], within the intervals $[1 - \tilde{\Lambda}_2, 1 - \tilde{\Lambda}_1]$, $[\tilde{\Upsilon}_1 - 1, \tilde{\Upsilon}_2 - 1]$ the samples $M_{1np} = M_1(l_n, v_p)$, $M_{2n} = M_1(l_n, 0)$, $S_{3np} = S_3(l_n, v_p)$ are calculated from the random fields $M_1(l, v)$, $M_2(l)$ (8) and deterministic function $S_3(l, v)$ (8). Here $l_n = 1 - \tilde{\Lambda}_2 + n\Delta l$, $n = \overline{0, \mathrm{int}\{(\tilde{\Lambda}_2 - \tilde{\Lambda}_1)/\Delta l\}}$. The discretization step for the variable $\tilde{t}$ is selected to be equal to $\Delta\tilde{t} = 0.05/\mu_{\min}$, and for the variables l and v – to $\Delta l = \Delta v = 0.001$. As a result, the relative mean square error of the stepwise approximation of the functional $M(l, v)$ (7) derived from the generated samples $M_{np} = M_{1np} - M_{2n} - S_{3np}$ does not exceed 10 %, when calculated according to the technique [13].

The normalized MLEs l_m, v_m (6) are determined by the numbers $n_{\max}$, $p_{\max}$ of the maximum sample M_{np} of the functional (7) as $l_m = 1 - \tilde{\Lambda}_2 + n_{\max}\Delta l$, $v_m = \tilde{\Upsilon}_1 - 1 + p_{\max}\Delta v$. Based on a series of the estimates obtained by processing of N realizations of the random field $M(l, v)$, where $N = 10^3$, the values of the sample biases and variances of the estimates l_m, v_m are calculated for the specified set of the parameters l_0, v_{02}, q, μ_1. Thus, with probability of 0.9 confidence interval boundaries deviate from experimental values no more than for 10...15 %.

In Fig.2., the theoretical dependences are drawn from the normalized conditional variance $V_l = V(\lambda_m|\lambda_0)/T^2$ of the estimate of the moment of abrupt change in the bandwidth λ_m (5) as the function of the normalized value q (9) of the spectral density of the random process $\xi(t)$ (1). Solid lines are calculated applying more accurate formulas (19)-(21), while dashed lines – asymptotic formulas (20), (22). The curves 1 are plotted for $v_{02} = 0.5$, $\mu_1 = 500$; 2 – $v_{02} = 0.5$, $\mu_1 = 1200$; 3 – $v_{02} = 0.75$, $\mu_1 = 1200$. The corresponding experimental values of the conditional variance of the moment of abrupt change in the bandwidth are designated by squares, crosses, and rhombuses. Here the true value of the parameter l_0 (10) is taken to be 0.5.

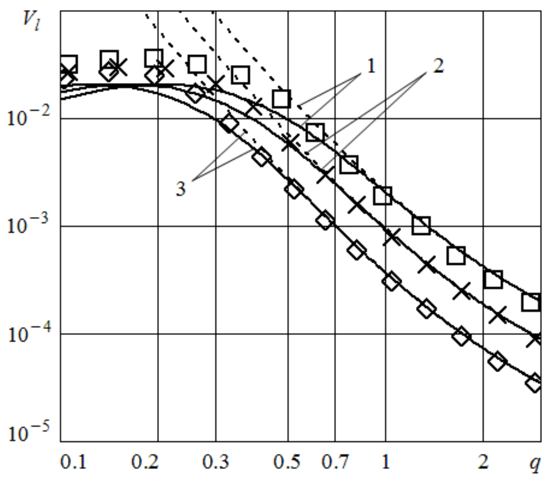

Fig.2. Normalized variance of the estimate of the moment of abrupt change in the bandwidth.

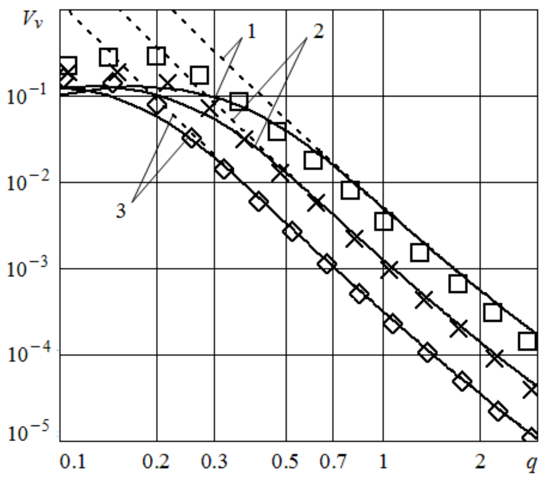

Fig.3. Normalized variance of the estimate of the bandwidth after the abrupt change.

In Fig.3., by solid and dashed lines the similar theoretical dependences are shown of the normalized conditional variance $V_v = V(\Omega_m|\Omega_{02})/\Omega_{01}^2$ of the estimate of the bandwidth after the abrupt change Ω_m (5) upon the parameter q, calculated according to (19)-(21) and (22). The curves 1 are obtained for $l_0 = 0.25$, $\mu_1 = 500$; 2 – $l_0 = 0.25$, $\mu_1 = 1000$; 3 – $l_0 = 0.5$, $\mu_1 = 1000$. The corresponding experimental values of the conditional variance of the abrupt change in the bandwidth are designated by squares, crosses, and rhombuses. In this case, the true value of the parameter v_{02} (10) is taken to be –0.5.

From the conducted analysis and Fig.2., Fig.3. it follows that the theoretical dependences obtained for the variances $V(\lambda_m|\lambda_0)$, $V(\Omega_m|\Omega_{02})$ (20), (21) already agree quite successfully with the experimental data, at least, under $\mu_1 \geq 100$, $q \geq 0.1$, $\tilde{\Lambda}_1 \geq 0.1$, $\tilde{\Lambda}_2 \leq 0.9$, $|\Omega_{02} - \Omega_{01}|/\Omega_{01} \geq 0.1$. And if output SNR is big enough, that is $z \geq 3$ (12), then the simpler approximations (22) can be used for calculating the variance of the estimate of the moment and the magnitude of the abrupt change of the random process bandwidth.

As it is noted in [10], for very big values of q the deviation of the experimental values of the variances $V(\lambda_m|\lambda_0)$, $V(\Omega_m|\Omega_{02})$ may be observed from the corresponding theoretical dependences obtained while using (20), (21) or (20), (22). It is the result of the formulas for the characteristics (11) of the functional $M(l,v)$ (7) having been found on the assumption that the sizes of order of the correlation time of the process $\xi(t)$ are negligible. Therefore, when the variances of MLEs λ_m/T, Ω_m/Ω_{02} decrease to the size of order $\mu_{\min}^{-2}$ (3), the calculation errors in (20)-(22) become considerable.

6. CONCLUSION

In order to identify the abrupt change point in the fast-fluctuating Gaussian process, the maximum likelihood method can be effectively applied. This approach allows us to obtain the algorithms for measuring the unknown moment of abrupt change and frequency parameter jumps of the random process, while neglecting the values of the order of its correlation time. These algorithms are technically the simplest ones in comparison with the common analogues. We apply the additive local Markov approximation method to write down the closed analytical expressions for the efficiency characteristics of the maximum likelihood measurer.

We used the statistical simulation to establish that the obtained theoretical results successfully agree with the corresponding experimental data in a wide range of the observable data realization parameter values. Additional researches show that the measurers synthesized by means of the introduced approach can also be used in the analysis of the non-Gaussian random processes with unknown piecewise constant parameters and bring no great losses in performance.

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