Time Delay Estimation of Stochastic Signals Using Conditional Averaging

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Abstract. *The results of the theoretical analysis of selected features of two statistical methods used in time delay estimation (TDE): direct cross-correlation (CCF) and the function of conditional average value of delayed signal (CAV) have been presented in this paper. The model of measured stochastic signals and principle of TDE using CAV and CCF are described in this paper. The relative uncertainty of both functions in extreme point and the relative standard deviations of TDE using CAV and CCF are evaluated and compared. The authors conclude that the method CAV described in this paper has less statistical errors in magnitude and location (time delay) estimation then CCF and can be applied to time delay measurement of random signals.*

Keywords: Time Delay Estimation, Random Signals, Conditional Averaging, Cross-Correlation

1. Introduction

The problem of time delay estimation *(TDE)* is significant in many areas as radar and sonar technology, radioastronomy, location of signal transfer paths or contact-free measurements of transport parameters. Determination of the time delay of stochastic signals received from two or more sensors is commonly carried out with the use of statistical methods. This problem has been thoroughly presented in the literature [1-6], which describes a well-known methods of signals analysis in the time and frequency domains. Among the traditional methods used for stationary signals, the most common one is direct cross-correlation (*CCF*) in the time domain and the phase of cross-spectral power density in the frequency domain [1-3, 7]. The methods based on conditional averaging of signals are relatively new [8-11].

This paper presents the results of comparative research of selected features *CCF* and the method which uses conditional averaging of the delayed signal (*CAV*) [11]. The relatively standard deviations of both functions in extreme points and standard deviations of *TDE* for *CCF* and *CAV* were evaluated and compared. The values of signal-to-noise ratio were determined for the assumed signal models, where the analysed methods had smaller standard deviations of estimation for specific parameters of the analysis.

2. Model of Measurement Signals and Principle of *TDE* Using *CCF* and *CAV*

In the case of many *TDE* applications (i.e. measurements of transport parameters of solids and flows), the relation for signals $x(t)$ and $z(t)$ received from two sensors is usually given by the following formula [1,3]:

$$z(t) = kx(t-\tau_0) + n(t) = y(t) + n(t), \tag{1}$$

where: $x(t)$ is the stationary random signal with a normal probability distribution $N(0, \sigma_x)$, frequency band B and the unilateral spectral power density:

$$G_{xx}(f) = \begin{cases} G & 0 < f \le B \\ 0 & f > B \end{cases}, \tag{2}$$

k, G are the constant factors; $\tau_0 = d/V$ is the transport delay equal to the quotient of the sensor spacing distance *d* and the average velocity of object *V*; *n(t)* is the stationary white noise, non-correlated with signal *x(t)*, with the distribution of $N(0, \sigma_n)$. The auto-correlation function of signal *x(t)* has the following form:

$$R_{xx}(\tau) = GB\left(\frac{\sin 2\pi B\tau}{2\pi B\tau}\right). \tag{3}$$

The direct cross-correlation $R_{xz}(\tau)$ of the signals described by the relation (1) can be expressed by formula [1]:

$$R_{xz}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_0^T x(t)z(t+\tau)dt \tag{4}$$

The function (4) achieves the maximum value for $\tau = \tau_0$, so that the delay can be determined as the argument of the main extreme of this function:

$$\tau_0 = arg\{max\, R_{xz}(\tau)\} = arg\{R_{xz}(\tau_0)\}. \tag{5}$$

The expected conditional value of the delayed signal *z(t)* for the condition $x(t) = x_p$ is defined as follows [11]:

$$A_z(\tau) = A_z\Big|_{x(t)=x_p}(\tau) = \int_{-\infty}^{\infty} z(t+\tau)p\left(z(t+\tau)\Big|_{x(t)=x_p}\right)dz(t+\tau), \tag{6}$$

where $p\left(z(t+\tau)\big|_{x(t)=x_p}\right)$ is the conditional probability density for the signal *z* value at the condition $x = x_p$, x_p – selected threshold value.

A good estimator of the expected conditional value (6) is the arithmetic conditional average value of the delayed signal. In practice, its determination entails detection of non-cross-correlated instant of threshold x_p transition of the original signal *x(t)*, starting the registration of the delayed signal *z(t)* fragments in those moments and averaging the set of their value.
Peak position of *CAV* determine the transport delay τ_0:

$$\tau_0 = arg\{max\, A_z(\tau)\} = arg\{A_z(\tau_0)\}. \tag{7}$$

The principle of time delay estimation based on *CCF* and *CAV* is shown in Figure 1.

a) b)

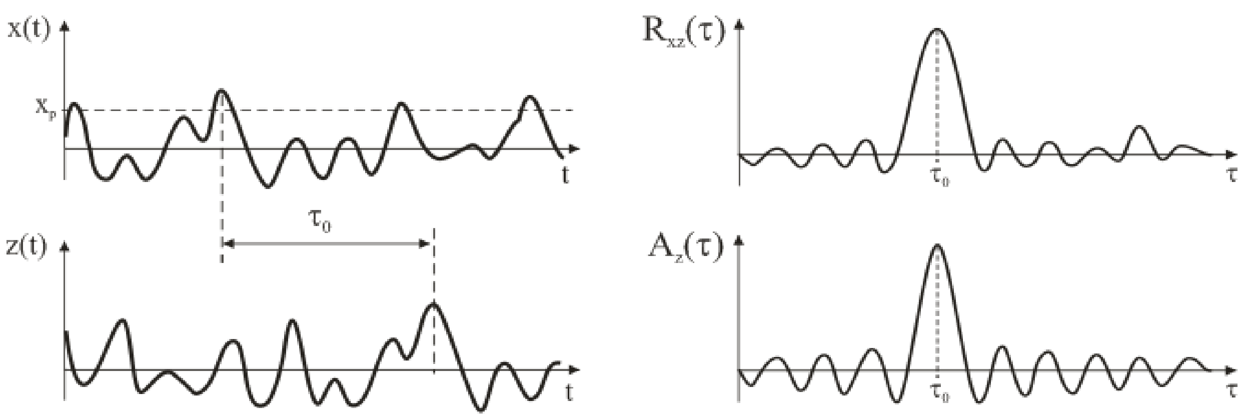

Fig. 1. The principle of time delay estimation using *CCF* $R_{xz}(\tau)$ and *CAV* $A_z(\tau)$: a) mutual delayed stochastic signals, b) *CCF* and *CAV* functions.

3. Comparison of Statistical Errors of *CCF* and *CAV* in Magnitude Estimates

The accuracy of *TDE* depends on gradients and variances of analysing function in the neighbourhood of the maximum. The relative standard uncertainty (relative standard deviation) of *CCF* can be written as [1,12]:

$$u_{rel}\left[\hat{R}_{xz}(\tau_0)\right]=\frac{\sigma\left[\hat{R}_{xz}(\tau_0)\right]}{R_{xz}(\tau_0)}=\frac{1}{\sqrt{N}}\sqrt{2+\frac{1}{k^2 SNR}} \tag{8}$$

where N – number of non-correlated samples of signals $x(t)$ and $z(t)$, $SNR =(\sigma_x/\sigma_n)^2$ – signal-to-noise ratio.

The relative standard uncertainty of *CAV* can be presented as [11]:

$$u_{rel}\left[\hat{A}_z(\tau_0)\right]=\frac{\sigma\left[\hat{A}_z(\tau_0)\right]}{A_z(\tau_0)}=\frac{1}{\sqrt{M}}\frac{\sigma_n}{kX_p}=\frac{1}{k\alpha\sqrt{M\cdot SNR}} \tag{9}$$

where M – number of non-correlated averaged segments of the delayed signal $z(t)$, $\alpha =(x_p/\sigma_x)$ – relative threshold value.

As the result of comparison of (8) and (9) the following expression is obtained:

$$\frac{u_{rel}\left[\hat{A}_z(\tau_0)\right]}{u_{rel}\left[\hat{R}_{xy}(\tau_0)\right]}=\frac{1}{\alpha}\left[\frac{M}{N}\left(2k^2 SNR+1\right)\right]^{-1/2} \tag{10}$$

Because *CCF* and *CAV* should be determined using non-correlated samples of signals the value *M/N* can be equal or less then 1. The relation $u_{rel}\left[\hat{A}_z(\tau_0)\right]/u_{rel}\left[\hat{R}_{xy}(\tau_0)\right]=f(SNR)$ for $M/N = 1$, $k = 1$ and selected values of α is presented in Figure 2. In this case the relative standard uncertainty of *CAV* is always less then for *CCF* if relative threshold value $\alpha \geq 1$ (Fig. 2).

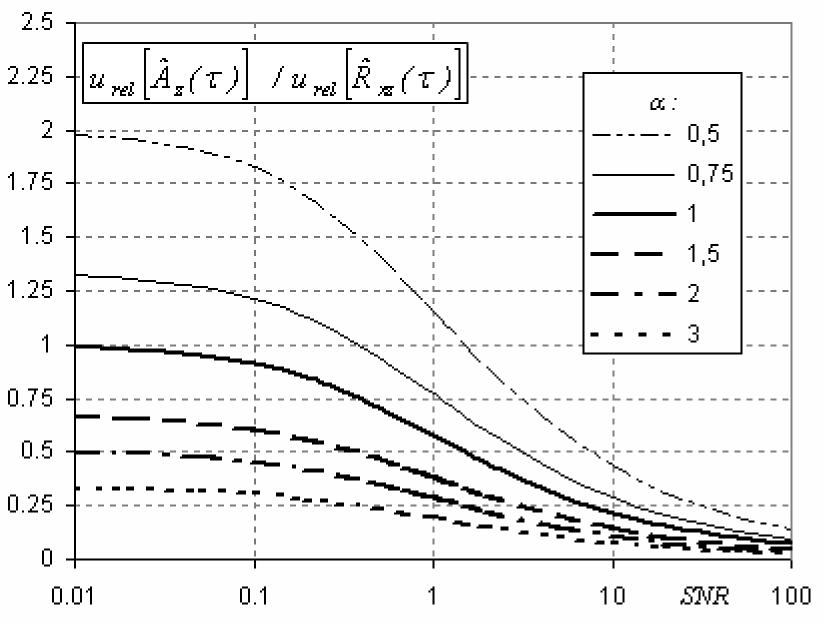

Fig. 2. The relation $u_{rel}\left[\hat{A}_z(\tau_0)\right]/u_{rel}\left[\hat{R}_{xy}(\tau_0)\right]=f(SNR)$ for $M/N = 1$, $k = 1$ and selected values of α

The results of work [11] shows, that optimum value of relative threshold value α is equal obout 2. The dependence $u_{rel}\left[\hat{A}_z(\tau_0)\right]/u_{rel}\left[\hat{R}_{xy}(\tau_0)\right]=f(SNR)$ for α = 2, *k* = 1 and selected values of *M/N* are presented in Figure 3.

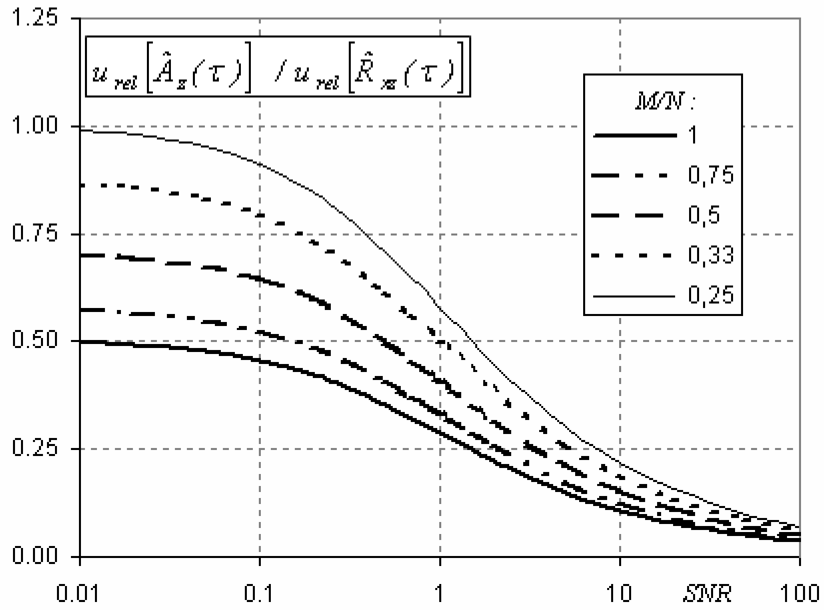

Fig. 3. The relation $u_{rel}\left[\hat{A}_z(\tau_0)\right]/u_{rel}\left[\hat{R}_{xy}(\tau_0)\right]=f(SNR)$ for α = 2, *k* = 1 and selected values of *M/N*

4. Comparison of Statistical Errors of *CCF* and *CAV* in Locations Estimates

The standard deviation of the transport delay τ_0 obtained by *CCF* based on [1,2] can evaluated to formula:

$$\sigma\left[\tau_0\right]_{CCF}=\frac{1}{\pi B}\sqrt[4]{\frac{3}{4}}\left\{\left(u_{rel}\left[\hat{R}_{xy}(\tau_0)\right]\right)^{1/2}\right\}^{1/2}= \\ =\frac{1}{\pi B}\sqrt[4]{\frac{3}{4}}\left\{\frac{1}{\sqrt{N}}\left[2+\frac{1}{k^2 SNR}\right]^{1/2}\right\}^{1/2} \tag{11}$$

For *CAV* can be written, respectively [11]:

$$\sigma\left[\tau_0\right]_{CAV}=\frac{1}{\pi B}\sqrt[4]{\frac{3}{4}}\sqrt{u_{rel}\left[\hat{A}_z(\tau_0)\right]}=\frac{1}{\pi B}\sqrt[4]{\frac{3}{4}}\frac{1}{\sqrt{k\alpha}}\left(M\cdot SNR\right)^{-1/4} \tag{12}$$

The comparison of (11) and (12) equations results the following equation:

$$\frac{\sigma\left[\tau_0\right]_{CAV}}{\sigma\left[\tau_0\right]_{CCF}}=\frac{1}{\sqrt{\alpha}}\left[\frac{M}{N}\left(2k^2 SNR+1\right)\right]^{-1/4} \tag{13}$$

The relation $\sigma[\tau_0]_{CAV}/\sigma[\tau_0]_{CCF}=f(SNR)$ for *M/N* = 1, *k* = 1 and selected values of α is presented in Fig. 4. The dependence (13) for α = 2, *k* = 1 and selected values of *M/N* are presented in Fig. 5. Similarly to magnitude estimation, standard deviation of transport delay for *CAV* is less than the corresponding values for *CCF* irrespective of the *SNR* values if relative threshold value $\alpha \geq 2$ and $M/N \geq 0{,}25$.

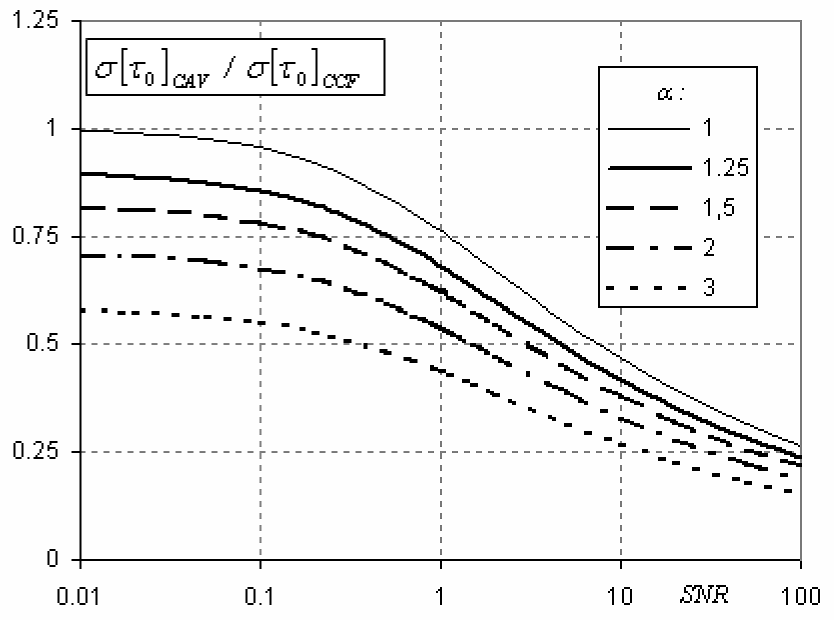

Fig. 4. The relation $\sigma[\tau_0]_{CAV} / \sigma[\tau_0]_{CCF} = f(SNR)$ for $M/N = 1$, $k = 1$ and selected values of α

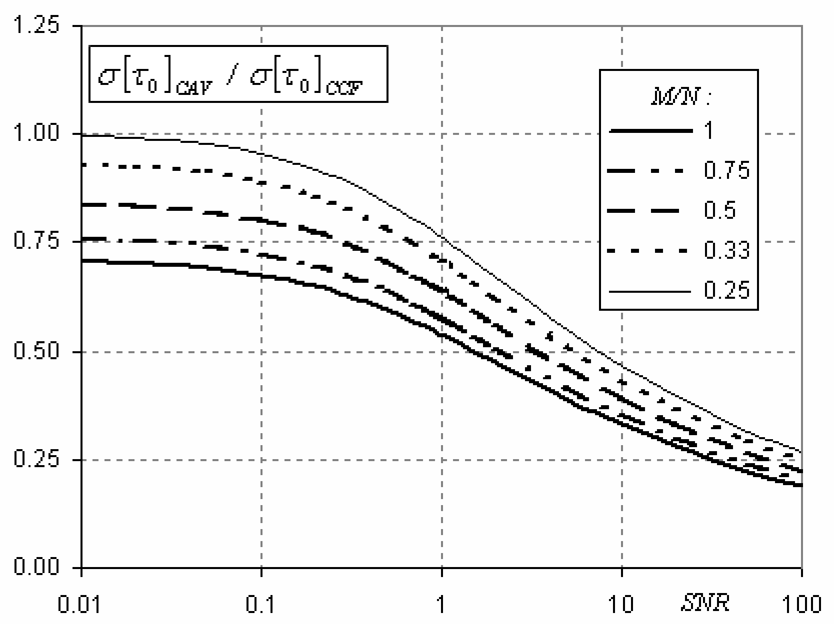

Fig. 5. The relation $\sigma[\tau_0]_{CAV} / \sigma[\tau_0]_{CCF} = f(SNR)$ for $\alpha = 2$, $k = 1$ and selected values of M/N

5. Conclusions

This work entailed the comparison of statistical errors of the direct cross-correlation and the conditional average value of the delayed signal in magnitude estimation and time delay estimation for the assumed signal models and the given *SNR* values. The theoretical analysis implies that the relative standard uncertainty of the *CAV* at the extreme points is less than the

corresponding standard uncertainty for *CCF* irrespective of the *SNR* values if $\alpha \geq 2$ and *N/M* is in the range $0{,}25 \leq N/M \leq 1$.
The standard deviation of transport delay obtained using *CAV* is less than the corresponding values for *CCF* independently of the *SNR* values if the relative threshold value $\alpha \geq 2$ and $0{,}25 \leq N/M \leq 1$.

The experimental verification of the theoretical analysis presented in this work is currently undergoing further investigation.

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