Stochastic and Deterministic Dithering in Slow Measurements

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Abstract. *Unified concept for theoretical analysis of dithering techniques is presented. Slow measurements are considered where the dither frequency and the sampling frequency are much higher than the signal frequency. Averaging of samples is engaged in the process of sampled signal filtering. The Widrow's statistical model is directly applied for stochastic dithering (SD) and characteristic functions are used for evaluation of error moments. Both nonsubtractive and subtractive form of SD is covered by the developed methodology. Simulation results are confronted with theoretical analysis. The performance of SD is also compared with the performance of deterministic dithering (DD).*

Keywords: Quantization, Widrow's model, Dithering, ADC errors

1. Introduction

Dither is usually a random noise added to a signal prior to it (re)quantization in order to control the statistical properties of the quantization error. In measurement applications the dithering process yields to suppression of quantization error by mean value filtering of the quantizer output. Two types of such stochastic dithering are distinguished. A term subtractive (stochastic) dithering (SD) is used for the case, when the dither is subsequently subtracted from the sampled signal after quantization. In the process of nonstubtractive (stochastic) dithering (ND) the dither is not subtracted from processed quantized signal. Both alternatives are commonly used and further developed e.g. for microcomputer based measurements [1]. The SD offers better performance especially for audio/video applications, while the implementation of ND is easier. Moreover special deterministic dithering technique exists based on deterministic signal added to the system input.

Several theoretical concepts were presented for analysis of dither influence on quantization process. Generally statistical methods are useful for investigation of stochastic dithering while Fourier analysis could be demanded for application of DD. The paper presents unified concept for analysis of dithering techniques based on Widrow's statistical theory of qunatization. The total root mean squared error is confronted for both alternatives – SD and ND. Furthermore, we will show that the statistical concept could be helpful in analysis of DD too. The impact of sampling non-coherency is investigated using Fourier analysis. The analysis should explain trends in error behavior related to the non-coherency.

2. Statistical Theory of Stochastic Dithering

In the system with nonsubtractive dithering the quantizer input is $u=s+d$, where s is measured value and d is the added noise. The transfer function $Q(s)$ of an ideal quantizer without dither comprises quantization error, which characteristic $Q_e(s)$ exhibits known saw-tooth shape with zero mean and peak-to-peak value of one quantization step q. In the system with ND the quantizer output is $v=Q(u)=Q(s+d)$. It is difficult to see positive contribution of the dither from the direct expression of the corresponding error $\varepsilon=Q_e(s+d)+d$. For measurement application statistical moments of error should be of interest. Reasonable approach for evaluation of error moments from characteristics functions (CF) was suggested by Widrow

[2]. From Widrow's concept of mathematic modeling of quantization (similar to standard modeling of signal sampling) it is possible to obtain CF of ε in the form

$$CF_{\varepsilon}(w,s)=\sum_{k=-\infty}^{\infty}\operatorname{sinc}\left(\frac{q(w+k\Psi)}{2}\right)CF_{\mathrm{d}}(w+k\Psi)e^{jk\Psi s} \tag{1}$$

where CF_{d} is CF of dither and $\Psi=2\pi/q$ is a constant. If CF of stochastic variable is known its moments could be evaluated by derivatives of the CF (σ_{d} is standard deviation of dither)

$$E[\varepsilon|s]=\frac{1}{j}\frac{dCF_{\varepsilon}(w)}{dw}\bigg|_{w=0}=\sum_{k=1}^{\infty}\frac{q(-1)^k}{\pi k}CF_{\mathrm{d}}\left(k\frac{2\pi}{q}\right)\sin\left(k\frac{2\pi}{q}s\right) \tag{2}$$

$$E[\varepsilon^2|s]=\frac{1}{j^2}\frac{d^2CF_{\varepsilon}(w)}{dw^2}\bigg|_{w=0}=\frac{q^2}{12}+\sigma_{\mathrm{d}}^2+\frac{q^2}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^{\mathrm{k}}}{k^2}\left(CF_{\mathrm{d}}(k\Psi)-k\Psi\, CF_{\mathrm{d}}^{\mathrm{d1}}(k\Psi)\right)\cos(k\Psi s) \tag{3}$$

The first moment (2) means the mean error $E[\varepsilon|s]$ and the second (3) is a mean squared error $E[\varepsilon^2|s]$. Also the variance is determined by (2) and (3) as it holds $Var[\varepsilon|s] = E[\varepsilon^2|s] - E^2[\varepsilon|s]$. Moments are dependent on the actual s. The overall measure of the error is obtained by integration of mean squared error within one quantization step [3]. From (2) one gets

$$\mu_{\varepsilon}^2=\frac{1}{q}\int_{-q/2}^{q/2}\left(E[\varepsilon|s]\right)^2 ds=\sum_{k=1}^{\infty}\frac{q^2}{2\pi^2k^2}CF_d^2\left(k\frac{2\pi}{q}\right) \tag{4}$$

Similarly the mean value within one quantization step could be used for evaluation of overall root mean squared error (squaring is already included in $E[\varepsilon^2|s]$)

$$RMS_{\varepsilon}^2=\frac{1}{q}\int_{-\frac{q}{2}}^{\frac{q}{2}}E[\varepsilon^2|s]ds=\frac{q^2}{12}+\sigma_d^2 \tag{5}$$

and of overall variance

$$\sigma_{\varepsilon}^2=\frac{1}{q}\int_{-q/2}^{q/2}Var[\varepsilon|s]ds=RMS_{\varepsilon}^2-\mu_{\varepsilon}^2=\frac{q^2}{12}+\sigma_d^2-\sum_{k=1}^{\infty}\frac{q^2}{2\pi^2k^2}CF_d^2\left(k\frac{2\pi}{q}\right) \tag{6}$$

Theoretical overall mean value (4), root mean squared value (5) and variance (6) of error ε are simply estimations of error moments of repeated measurements, if the probability of every position of s relative to boundaries of a quantization bin is the same.

If the mean error $E[\varepsilon|s]$ is evaluated for dithering systems and further investigated, it reveals that this mean has lower peak-to-peak value compared to original error characteristic $Q_{\mathrm{e}}(s)$ of the quantizer. This is the usual principle of dithering usage for suppression of quantization error for measurement applications. To achieve the error suppression the mean value has to be estimated from the samples employing low pass filter assuming constant measured value s during one measurement with the mean estimation. Unfortunately the filter output includes a random part for finite number of samples. Averaging is usual type of filter for dithering applications. The variance of the error after averaging $\xi = o\text{-}s$ is suppressed to $\sigma_{\varepsilon}^2=\sigma_{\varepsilon}^2/N$. Therefore also RMS of the final error ξ of system with ND is smaller compared to error (5) without dithering. It could be evaluated from (4) and (6) as

$$RMS_{\xi}^2(N)=\mu_{\varepsilon}^2+\frac{\sigma_{\varepsilon}^2}{N}=\frac{\frac{q^2}{12}+\sigma_{\mathrm{d}}^2}{N}+\left(1-\frac{1}{N}\right)\sum_{k=1}^{\infty}\frac{q^2}{2\pi^2k^2}CF_{\mathrm{d}}^2\left(k\frac{2\pi}{q}\right) \tag{7}$$

Evidently, for N = 1 the error expression simplifies to equation (5) and for $N\rightarrow\infty$ to (4). For some dither types and sufficiently large variance σ_d^2 considering only the first term of the sum in (7) could still lead to precise estimation of error [4].

Subtractive Dithering

Results of previous ND analysis could be easily adopted for system with SD and to its error ξ_{SD}. Mean error (4) remains the same (for symmetrical dither PDF with zero mean). The RMS error does not contain dither itself, as it was subtracted after quantization. Finally we can write

$$RMS_{\xi SD}^2(N) = \frac{q^2}{12N} + \left(1 - \frac{1}{N}\right)\sum_{k=1}^{\infty}\frac{q^2}{2\pi^2 k^2}CF_d^2\left(k\frac{2\pi}{q}\right) \tag{8}$$

For properly chosen dither distribution and variance the sum in (8) is zero, when the theoretical improvement of quantizer resolution is simply $\log_2(\sqrt{N})$ bits.

3. Deterministic Dithering with Non-Coherent Sampling

Deterministic signal could be used similarly as stochastic one for improvement of quantizer resolution. However the determinism of dither requires special access to certain features and regularities. For slowly changing measured value the achieved precision then could be estimated by formulas obtained for stochastic ND as DD is commonly implemented as nonsubtractive. The correlation between sampling and dither combined with averaging allows suppression of dither from samples in the extent that it reproduces subtractive stochastic dithering [5].

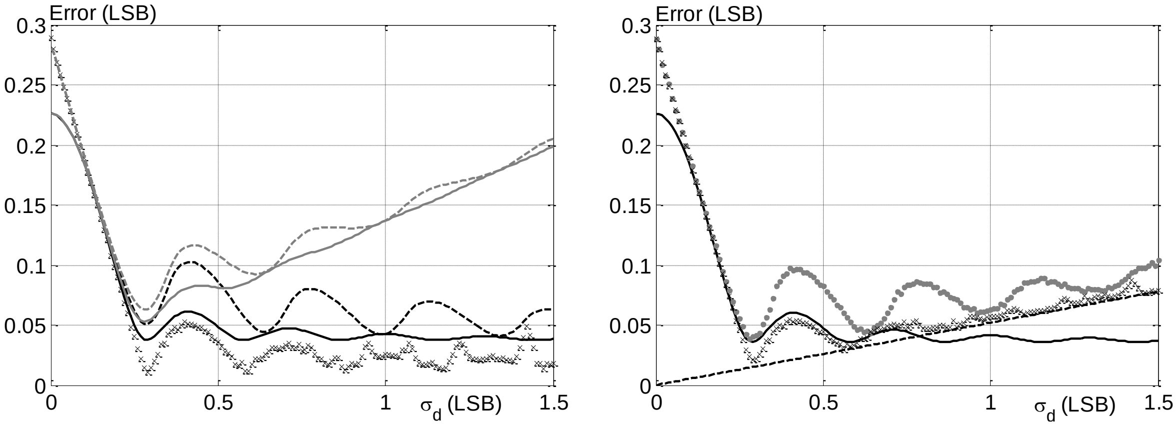
For correlated sampling of dither the best case is to achieve coherent sampling, when the samples correspond to integer multiples of dither periods. For non-coherent sampling the mean value estimated from samples, like any spectral component, is loaded by residual error. For its investigation Fourier analyses can be employed. Generally n-th spectral component of error is

$$FR_{\mathrm{DDE},n} = \frac{1}{T_w}\sum_{k=-\infty}^{\infty} FS_{d,k}FT_w\left(n\omega_w - k\omega_d\right) \tag{9}$$

where $FS_{d,k}$ is the k-th coefficient of Fourier series of deterministic dither and FT is Fourier transform of considered rectangular window.

4. Results and Discussion

Results of theoretical analyses were confronted with simulations and all discussed types of dither were compared together. In Fig.1 errors ξ for ND, ξ_{SD} for SD and $FR_{\mathrm{DDE},0}$ for DD are depicted in dependence on standard deviation σ_d of dither. Uniform and sinusoidal distributions of stochastic dithers are considered. For every simulated level of σ_d the RMS of error was evaluated from 20 repeated measurements of s uniformly distributed within one quantization step. For stochastic dithering theoretical error values are close to simulation results (except the area of very small σ_d) therefor only theoretical curves are plotted. However for sinusoidal distribution simplification of sums in (7) and (8) to only one member would cause significant deviation of theoretical estimations especially near the minimums and therefore 10 members where used. From presented graphs one can observe better performance of uniform dither against sinusoidal for both ND and SD and evidently better accuracy of SD for optimal or large σ_d. However DD still offers higher accuracy for coherent sampling (sampled exactly 3 periods here) of triangular waveform. Note that the shape of the error curve for DD is similar to theoretical curve of SD with uniform distribution but the error is lower for DD.



a) b)

Fig. 1. Error dependence on dither standard deviation: a) ND (black) and SD (grey) with uniform (solid) and sinusoidal (dashed) distribution of dither compared with DD ("x") witch triangular waveform; b) DD with triangular (black "x") and cosine (grey "•") waveform confronted with theory of SD (for uniform dither - solid curve) and with theory of non-coherent sampling (for triangular waveform - dashed line).

The results of the DD behavior analysis by non-coherent sampling is shown in Fig.1b. We can see that despite higher number of samples the accuracy is worse than in the previous case. While there are still fluctuations correlated with the shape of theoretical curve of SD with rising σ_d the error dependency tends to theoretical error of non-coherent sampling (9). The results were plotted also for cosine waveform. Again the shape is similar to curve of SD with sinusoidal distribution (see Fig.1a) while the trend is rising because of non-coherent sampling.

Finally we can claim that for measurement of slow varying signals deterministic dithering leads to better accuracy compared to even subtractive stochastic dithering if conditions close to coherent sampling of dither are achieved. For the estimation of the shape of the RMS error curves the presented theory of stochastic dithering offers precise tool for both subtractive and non-subtractive form of stochastic dithering and approximate tool for deterministic dithering.

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