Fiducial Generalized Pivots for a Variance Component vs. an Approximate Confidence Interval

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Abstract. The paper is devoted to comparison of different confidence intervals for a variance component corresponding to the random factor in mixed linear models with two variance components. Namely, generalized confidence intervals (based on fiducial generalized pivots (FGP), see Hannig et al. (2006)) are compared to an approximate solution suggested by El-Bassiouni (1994). We focus on FGPs of certain types that were shown to yield exact intervals in some limiting situations. This makes them more equivalent counterparts to the El-Bassiouni's interval, if frequentist properties are the main concern. A simulation study is presented that reveals, how besides sharing the limiting properties, these FGPs and the mentioned approximate interval perform with respect to each other. We also comment on a relationship between the considered types of FGPs.

Keywords: mixed linear model, fiducial generalized pivots, variance components, approximate confidence intervals

1. Introduction

Construction of confidence intervals for a variance component corresponding to the random factor in mixed linear models with two variance components has been studied extensively in literature. As the problem is complicated by the presence of a nuisance parameter - the unknown variance of error, suggested solutions are either approximate or based on fiducial generalized pivots (FGP) (as defined in [1]). Recently attention has been focused especially on the latter ones. Properties of the resulting generalized confidence intervals (GCI) were examined in simulation studies e.g. in [3, 4, 5]. (In general, the confidence level of a GCI cannot be guaranteed and is usually checked only by simulations.) In the studies the GCIs were also compared to some previously suggested approximate intervals, however, the one proposed by El-Bassiouni in [2] has never been considered. The main purpose of this paper is to fill in this gap, as El-Bassiouni's interval seems to be a solid representative of the approximate solutions. With respect to maintaining the nominal confidence level, it can be recommended for general use (without further knowledge of the true magnitude of the parameters) and besides performing well in simulations, it can be shown to behave as an exact interval in 3 limiting situations (see e.g. [6]). The competing GCIs considered in this paper are based on FGPs of two types that share the three limiting properties with El-Bassiouni's interval (see [7]), which makes them more acceptable from the frequentist point of view. In the conducted simulation study the mentioned GCIs and the approximate El-Bassiouni's interval are compared with respect to their probability of coverage and average length. Besides, the study illustrates the relationship between the considered FGPs and in some cases it supplements results in [8] where the respective GCIs were examined only by means of two examples. The paper is organized as follows: the results are stated in section 3, while the model, the used FGPs and the approximate interval are described in section 2.

2. Subject and Methods

Model

The vector of observations y $(n \times 1)$ is supposed to come from a multivariate normal distribution $N_n(Xb, \sigma_1^2 ZZ' + \sigma^2 I)$, where X, Z are known matrices (we suppose that $\mathcal{R}(Z) \nsubseteq \mathcal{R}(X)$, where $\mathcal{R}(A)$ denotes the linear subspace generated but the columns of the matrix A) and b, $(\sigma_1^2,\ \sigma^2)'$ are vectors of unknown parameters, $\sigma_1^2 \geq 0$, $\sigma^2 > 0$. Exploiting the principle of invariance, inference on variance components σ_1^2, σ^2 is based on a minimal sufficient statistic for a maximal invariant under translation in mean, which is a vector consisting of mutually independent quadratic forms: $U_i \sim (\lambda_i \sigma_1^2 + \sigma^2)\chi^2_{\nu_i}$, $i = 1, ..., r$ $(\lambda_1 > ... > \lambda_{r-1} > \lambda_r \geq 0)$. See also [5]. We suppose that $\lambda_r = 0$. This is fulfilled whenever $n > rank([X,\ Z])$. The model is then said to have non-zero degrees of freedom for error as $U_r \sim \sigma^2 \chi^2_{\nu_r}$. Construction of confidence intervals on σ_1^2 is complicated by the presence of the nuisance parameter σ^2.

In what follows, $F_{m,n;x}$ and $\chi^2_{m;x}$ denote x quantiles of the appropriate F and χ^2 distributions and $s = \sum_{i=1}^{r-1} \nu_i$.

Approximate solution

In [2] El-Bassiouni proposed the following approximate confidence interval that maintained the nominal confidence level in all considered settings. Moreover, its probability of coverage tends to that of an exact interval when $\nu_r \to \infty$, or $\sigma_1^2/\sigma^2 \to \infty$, or $\sigma_1^2 = 0$ (see e.g. [6]). Its bounds are non-negative solutions (or zeros if such solutions do not exist) to the following equation in B :

$$\sum_{i=1}^{r-1} \frac{U_i}{\lambda_i B \dfrac{\chi^2_{s;x}}{sF_{s,\nu_r;x}} + \dfrac{U_r}{\nu_r}} = sF_{s,\nu_r;x}, \tag{1}$$

where $x = 1 - \alpha/2$ for the lower and $x = \alpha/2$ for the upper bound. (Forcing the bounds to be non-negative stems from the fact that σ_1^2 is a non-negative parameter.) The interval is a generalization (for $r > 2$) of the Williams-Tukey interval [10, 11], see [6]. Thus we will refer to it as the El-Bassiouni - Williams - Tukey interval (EBWT). In [2] a short version of the interval was also suggested, using not-equal-tailed χ^2 quantiles, like those in Table 678 in [12], while still using $x = 1 - \alpha/2$, $\alpha/2$ in F quantiles. In what follows, these short intervals will be referred to as sEBWT.

Fiducial generalized pivots

We recall a definition of an FGP given in [1]: Denote by $U^* = (U_1^*, ..., U_r^*)'$ an independent copy of $U = (U_1, ..., U_r)'$. An FGP is a function $R(U, U^*, \sigma_1^2, \sigma^2)$ with the following properties:

1. Conditional distribution of R, conditional on $U = u$ is free of σ_1^2, σ^2.

2. For every allowable u, $R(u,\ u,\ \sigma_1^2,\ \sigma^2) = \sigma_1^2$.

For more details see [1]. Then the GCIs are formed by appropriate quantiles (usually $\alpha/2$, $1 - \alpha/2$) of the conditional distribution of R conditionally on $U = u$.

Ex.	X	Z	s	ν_r
1	$[1_{30}\ v_{30}]$	n_i :5, 10, 15	2	26
2	$[1_{30}\ t]$ $t:\ 30\times 1$ $t_i=-3+6*(i-1)/29$	$[diag(1_6,\ 1_6,\ 1_6,\ 1_6,\ 1_6)\ w]$ $w: 30\times 1$ $w_i=(-2+4*(i-1)/29)^2$	5	23
3	$[1_{157}\ v_{157}]$	n_i :1, 1, 2, 3, 50, 100	5	150
4	1_{14}	n_i : 1, 1, 1, 1, 1, 1, 2, 2, 2, 2	9	4
5	$[1_{59}\ v_{59}]$	n_i : 1, 1, 4, 5, 6, 6, 8, 8, 10, 10	9	48

Table 1: Examples considered in the simulation study. v_k is a $(k\times 1)$ vector of real numbers between 0 and 1, 1_k is a $(k\times 1)$ vector of ones. If not stated otherwise, Z is a block matrix with 1_{n_i} on the diagonal. Ex. 1,3,5 were considered also in [3].

Denote $Q_i = U_i^*/(\lambda_i\sigma_1^2+\sigma^2)$, $i=1,...,r-1$ and $V=U_r^*/\sigma^2$. The type 1 FGPs considered in this paper are of the form:

$$R_1 = max(0,R),\ \text{where } R \text{ is a solution to the following equation}$$
$$\sum_{i=1}^{r-1}\frac{c_iU_i}{\lambda_iR+U_r/V}=\sum_{i=1}^{r-1}c_iQ_i, \tag{2}$$

and c_is are some positive constants. In particular, we take $c_i=1$ or $c_i=\lambda_i$, $i=1,..,r-1$, the corresponding FGPs being denoted R_1^1 , R_1^λ. R_1^1 was firstly considered in [3]. The choice of c_is was inspired by tests of $H_0:\ \sigma_1^2=0$. For $c_i=1$, $i=1,...,r-1$, 0 is included in the interval in accordance with the Wald test, which is optimal for large alternatives. The other choice of c_is correspond to a test that should be more powerful for small alternatives, see [13].

The type 2 FGPs considered in this paper are of the form:

$$R_2 = max(0,\tilde{R}),$$
$$\tilde{R}=\frac{\sum_{i=1}^{r-1}c_iU_i-\frac{U_r}{V}\sum_{i=1}^{r-1}c_iQ_i}{\sum_{i=1}^{r-1}c_i\lambda_iQ_i}, \tag{3}$$

and c_is are some positive constants. We take $c_i=1$ or $c_i=1/\lambda_i$, $i=1,..,r-1$, the corresponding FGPs being denoted R_2^1 , $R_2^{1/\lambda}$. GCIs based on these FGPs were suggested in [8] (there they were derived using generalized test variables T_1^1, $T_1^{1/\lambda}$).

Simulation study

Designs used in the simulation study comparing the different GCIs and the approximate intervals are stated in Table 1. For each design and $\sigma_1^2=0.001,\ 0.1,\ ...,\ 0.9,\ 0.999$, $\sigma^2=1-\sigma_1^2$, 2500 realizations of y were simulated and intervals using Eq. 1, 2, 3 were constructed for $\alpha=0.05$. Substituting $T=B\nu_r\chi^2_{s;x}/(sF_{s,\nu_r;x}Ur)$ and multiplying both sides by U_r/ν_r, Eq. 1 was solved in T by the Newton-Raphson method with tolerance 10^{-14} and then B was obtained by the reverse transformation. In case of GCIs, for each realized U_is, we randomly generated Q_is , V 10 000 times, solved Eq. 2 by the Newton-Raphson method with tolerance 10^{-12} and used these 10 000 values to estimate the quantiles (2.5% and 97.5%) of the conditional distribution of R_1^1 (R^λ), or in case of type 2 FGPs we used the generated Q_is and V for computing the value of R_2^1 ($R_2^{1/\lambda}$) and then we estimated the respective quantiles (2.5% and 97.5%).

3. Results and discussion

In Figure 1 and 2 the obtained simulated probabilities of coverage and average lengths of the GCIs and the approximate intervals are displayed for each example in Table 1. In Figure 1 there are depicted also bounds between which the estimated probability of coverage should fall with probability 0.95 if the true probability of coverage is 0.95, using the normal approximation to the binomial distribution. In three cases the obtained coverage probabilities fell below the lower bound of this interval, but only slightly and so we conclude that the nominal confidence coefficient seems sufficiently maintained by all considered intervals in all considered settings. We can also see that the approximate intervals tend to be a bit conservative in Ex. 4 ($s > \nu_r$) for small and medium σ_1^2/σ^2 and unlike the other intervals, which become exact with increasing ν_r (see [6, 7]), in Ex. 3 sEBWT remains a bit conservative for small values of σ_1^2/σ^2.

Comparing average lengths, we can observe the following

1. FGP R_1^λ yielded shorter intervals (on average) than R_1^1 for small values of σ_1^2/σ^2, while for larger ratios σ_1^2/σ^2, R_1^1 outperformed it in this regard. The difference is most striking in Ex. 2, which may be a result of a combination of a smaller ν_r and a wide range of λ_is (53.563, 6.0, 1.029, 0.233).

2. The average lengths of the approximate intervals, EBWT, and the GCIs based on R_1^1 are comparable, though, somewhat bigger (relatively to average length) differences between them appeared in Ex. 4 ($s > \nu_r$), with intervals based on R_1^1 shorter on average.

3. The short version of the approximate interval, sEBWT, was naturally shorter than EBWT intervals, and on average also shorter than GCIs based on R_1^1 in all considered settings. In Ex. 1 the average length of sEBWT was even the shortest of all compared intervals. For this design ($s = 2$) also the most dramatic difference between EBWT and sEBWT was observed. (A similar result had been obtained in [2].)

4. Similar results yielded by R_1^1 and EBWT suggest that by a different choice of quantiles in the R_1^1 procedure (not $\alpha/2$, $1 - \alpha/2$) we can perhaps obtain shorter GCIs with average length comparable to that of sEBWT.

5. Looking at pairs R_1^1, $R_2^{1/\lambda}$ and R_1^λ, R_2^1, their average lengths are very similar and seem to become the same as σ_1^2/σ^2 increases. (This was previously observed in [5] for the former pair.) Actually, it can be shown that normed by the true value of σ_1^2, the mean lengths of GCIs based on type 1 FGPs with constants c_i and based on type 2 FGPs with constants c_i/λ_i converge to the same quantity as $\sigma_1^2/\sigma^2 \to \infty$. For a brief outline of the proof see Appendix. We just note here that while $R_1^1,\ R_2^{1/\lambda}$ behave similarly from the viewpoint of average length, intervals based on the two FGPs may differ substantially with respect to other properties. For example, considering the ties of the FGPs to the tests of $H_0 :\ \sigma_1^2 = 0$, it is not surprising that the two intervals differ for small and moderate σ_1^2/σ^2 with respect to the probability of falsely including zero. (Just for an illustration: in Ex. 3 in our simulation 0 was included in 44%, 12.8%, 1.6% of intervals constructed by R_1^1, while in 92%, 61.2% and 11.7% of intervals constructed using $R_2^{1/\lambda}$ for $\sigma_1^2/\sigma^2 = 1/4,\ 1,\ 4$ respectively.)

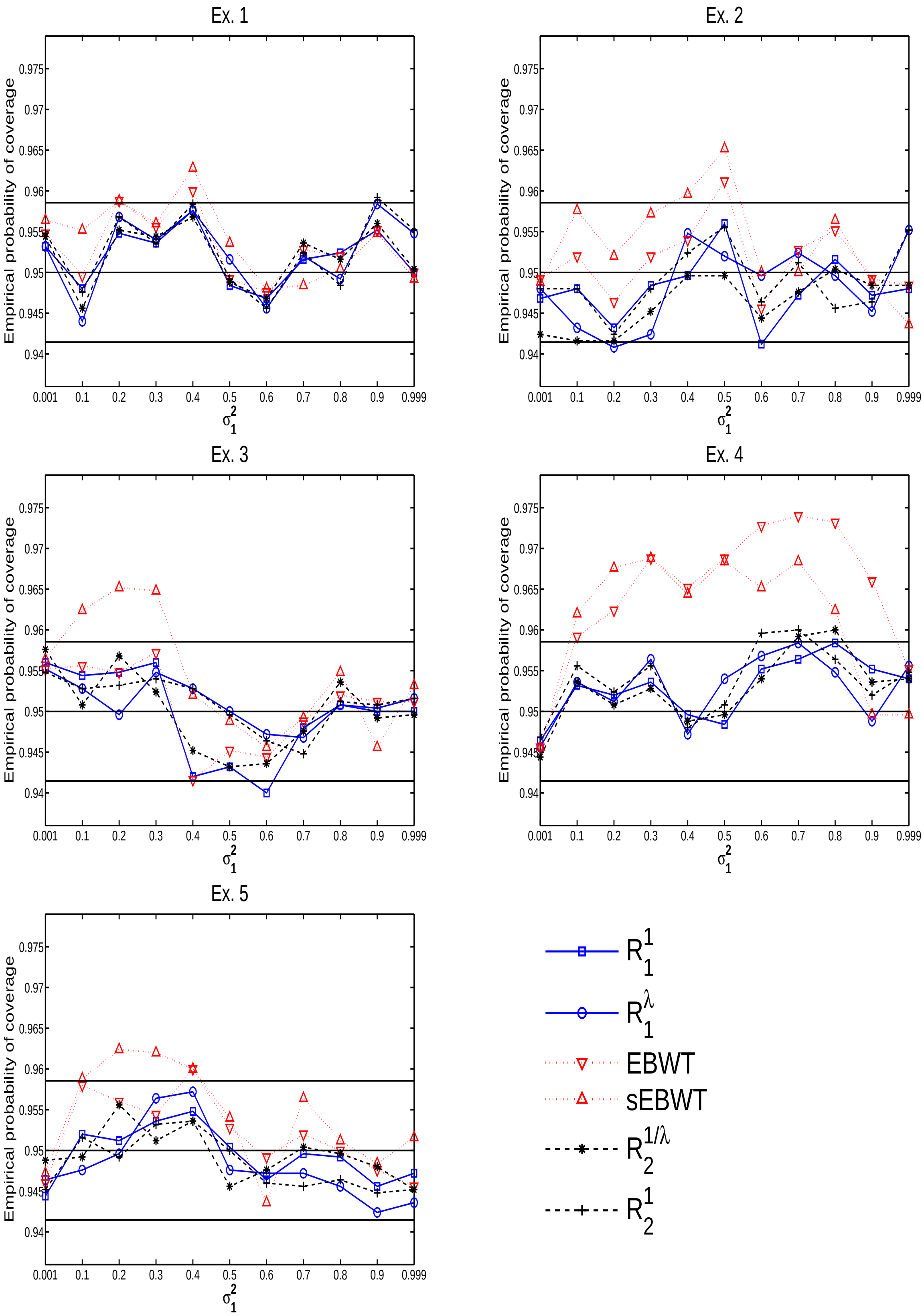

Figure 1: Simulated probabilities of coverage of the considered confidence intervals in examples from Table 1 (for the true values of variance components: σ_1^2, $\sigma^2 = 1 - \sigma_1^2$).

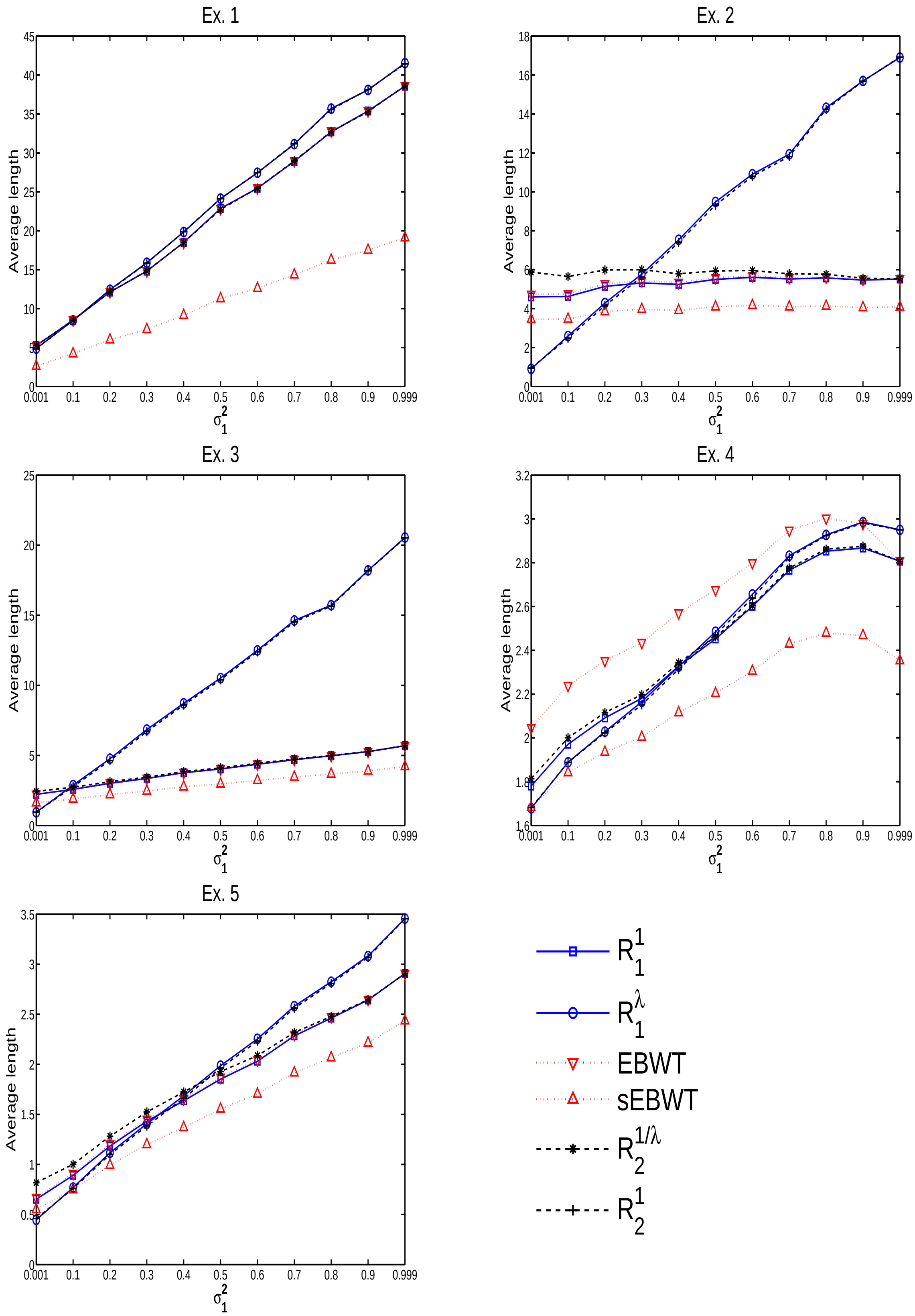

Figure 2: Average lengths of the simulated confidence intervals in examples from Table 1 (for the true values of variance components: σ_1^2, $\sigma^2 = 1 - \sigma_1^2$).

4. Conclusions

The paper deals with comparison of a few procedures for constructing confidence intervals for a variance component corresponding to the random factor in mixed linear models with two variance components. By means of simulation generalized confidence intervals based on certain fiducial generalized pivots (FGP) (of two types) were compared to an approximate interval, and its short version, suggested in [2] (called El-Bassiouni-Williams-Tukey interval here). All the considered procedures seemed to maintain the nominal confidence coefficient to a sufficient degree, thus they have been compared with respect to average lengths of the intervals they yielded. None of the procedures yielded the shortest intervals across all designs and values of the unknown parameters, though, if $s > 2$, both approximate solutions and the FGPs R_1^1 (an FGP firstly suggested in [3]) and $R_2^{1/\lambda}$ produced intervals, whose average lengths seem overall acceptable. Out of these the shortest intervals were obtained using the short version of the El-Bassiouni-Williams-Tukey interval. In case $s = 2$, this procedure yielded the shortest intervals of all. Also a certain relationship between the used types of FPGs has been commented on.

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Appendix

Here we present a brief proof that normed by the true value of σ_1^2, the mean lengths of a GCI constructed by R_1 with $c_i = d_i > 0,\ i = 1, ..., r-1$ and of a GCI constructed by R_2 with $c_i = d_i/\lambda_i,\ i = 1, ..., r-1$ converge to the same quantity.

Firstly, consider GCIs constructed by R_2 as defined in (3). For each observed value of $u = (u_1, ..., u_r)'$ a GCI is formed by the lower and upper quantiles of the conditional distribution of R_2 conditionally on u. Denote these $q_2^{\alpha/2}(u),\ q_2^{1-\alpha/2}(u)$. The mean length of the GCI normed by σ_1^2 is

$$E\left(\frac{q_2^{1-\alpha/2}(U)}{\sigma_1^2} - \frac{q_2^{\alpha/2}(U)}{\sigma_1^2}\right).$$

Thus let us consider the limiting behaviour of $E(q_2^{\alpha}(U)/\sigma_1^2)$ for fixed α as $\sigma_1^2/\sigma^2 \to \infty$. Note that $q_2^{\alpha}(u)/\sigma_1^2$ is an α quantile of $R_2(u, U^*, \sigma_1^2, \sigma^2)/\sigma_1^2$. Without loss of generality, we may write

$$\frac{U_i}{\lambda_i \sigma_1^2} = \left(1 + \frac{\sigma^2}{\lambda_i \sigma_1^2}\right) Z_i,\ i = 1, ..., r-1, \text{ and } \frac{U_r}{\sigma_1^2} = \frac{\sigma^2}{\sigma_1^2} Z_r,$$

where $Z_i \sim \chi^2_{\nu_i},\ i = 1, ..., r$ are mutually independent and independent of U^*. Then as $\sigma_1^2/\sigma^2 \to \infty$, for each $\omega \in \Omega$ (where (Ω, S, P) is the underlying probability space)

$$\frac{U_i(\omega)}{\lambda_i \sigma_1^2} \to Z_i(\omega),\ i = 1, ..., r-1 \text{ and } U_r(\omega)/\sigma_1^2 \to 0 \tag{4}$$

So for each observed u ($u = U(\omega)$ for some ω), using (3),

$$\frac{R_2(U(\omega), U^*, \sigma_1^2, \sigma^2)}{\sigma_1^2} = max(0, \tilde{R}/\sigma_1^2) \longrightarrow T_2(\omega) = \frac{\sum_{i=1}^{r-1} c_i Z_i(\omega) \lambda_i}{\sum_{i=1}^{r-1} c_i \lambda_i Q_i} \quad \text{in distribution}$$

Denoting $q_{1-\alpha}$ the $1-\alpha$ quantile of $\sum_{i=1}^{r-1} c_i\lambda_i Q_i$ and realising that for each ω, $T_2(\omega)$ is a random variable with a continuous distribution function, from the previous we have

$$q_2^{\alpha}(U)/\sigma_1^2 \to \sum_{i=1}^{r-1} c_i Z_i \lambda_i / q_{1-\alpha} \quad \text{for each } \omega.$$

Now, if in $\lim_{\sigma_1^2/\sigma^2\to\infty} E(q_2^{\alpha}(U)/\sigma_1^2)$ the order of the limit and expectation can be exchanged, for the mean length we obtain:

$$E\left(\frac{q_2^{1-\alpha/2}(U)}{\sigma_1^2} - \frac{q_2^{\alpha/2}(U)}{\sigma_1^2}\right) \to E\left(\frac{\sum_{i=1}^{r-1} c_i Z_i \lambda_i}{q_{\alpha/2}}\right) - E\left(\frac{\sum_{i=1}^{r-1} c_i Z_i \lambda_i}{q_{1-\alpha/2}}\right) = \frac{\sum_{i=1}^{r-1} c_i \nu_i \lambda_i}{q_{\alpha/2}} - \frac{\sum_{i=1}^{r-1} c_i \nu_i \lambda_i}{q_{1-\alpha/2}}. \quad (5)$$

Justification for exchanging the oder of the limit and integration comes from the following: for each ω and $\sigma_1^2/\sigma^2 > 1/\lambda_{r-1}$, $\frac{U_i(\omega)}{\lambda_i\sigma_1^2} \le 2Z_i(\omega)$ (for $i = 1, ..., r-1$), which implies that for each ω and $\sigma_1^2/\sigma^2 > 1/\lambda_{r-1}$, the cumulative distribution function of $2T_2(\omega)$ is stochastically greater or equal to the cumulative distribution function of $R_2(U(\omega), U^*, \sigma_1^2, \sigma^2)/\sigma_1^2$, thus for each ω and $\sigma_1^2/\sigma^2 > 1/\lambda_{r-1}$, $(0 \le) q_2^{\alpha}(U)/\sigma_1^2 \le 2\sum_{i=1}^{r-1} c_i Z_i \lambda_i / q_{1-\alpha}$. And the expectation of the latter exists.

Now, consider GCIs based on R_1. We are interested in the limit of

$$E\left(\frac{q_1^{1-\alpha/2}(U)}{\sigma_1^2} - \frac{q_1^{\alpha/2}(U)}{\sigma_1^2}\right),$$

which is a difference of the lower and upper quantiles of $R_1/\sigma_1^2 = max(0, R/\sigma_1^2)$ as defined in (2). R is an implicitly stated function $R(U_1c_1, ..., U_{r-1}c_{r-1}, \sum_{i=1}^{r-1} c_i Q_i, U_r/V)$ and it can be easily seen that $R/\sigma_1^2 = R\left(\frac{U_1c_1\lambda_1}{\sigma_1^2\lambda_1}, ..., \frac{U_{r-1}c_{r-1}\lambda_{r-1}}{\sigma_1^2\lambda_{r-1}}, \sum_{i=1}^{r-1} c_i Q_i, \frac{U_r}{\sigma_1^2 V}\right)$. From Implicit Function Theorem it follows that $R(\cdot)$ is continuous at every $(r+1)-$tuple with all but the last coordinate positive, the last one being zero. For each such an $(r+1)-$tuple $R(a_1, ..., a_{r-1}, a_r, 0) = \frac{\sum_{i=1}^{r-1} a_i/\lambda_i}{a_r}$. Continuity of $R(\cdot)$ together with (4) implies that for each observed $u = U(\omega)$ as $\sigma_1^2/\sigma^2 \to \infty$,

$$R_1/\sigma_1^2 \longrightarrow \sum_{i=1}^{r-1} c_i Z_i(\omega) / \sum_{i=1}^{r-1} c_i Q_i \quad \text{in distribution.}$$

Thus $q_1^{\alpha}(U)/\sigma_1^2$ converges for each ω to $\sum_{i=1}^{r-1} c_i Z_i/\tilde{q}_{1-\alpha}$, where $\tilde{q}_{1-\alpha}$ is an $(1-\alpha)$ quantile of $\sum_{i=1}^{r-1} c_i Q_i$. Similarly to the previous case, it can be shown that for each ω and $\sigma_1^2/\sigma^2 > 1/\lambda_{r-1}$, $(0 \le) q_1^{\alpha}(U)/\sigma_1^2 \le 2\sum_{i=1}^{r-1} c_i Z_i/\tilde{q}_{1-\alpha}$ and as the expectation of the latter exists, for the mean length normed by the true σ_1^2 we obtain:

$$E\left(\frac{q_1^{1-\alpha/2}(U)}{\sigma_1^2} - \frac{q_1^{\alpha/2}(U)}{\sigma_1^2}\right) \to E\left(\frac{\sum_{i=1}^{r-1} c_i Z_i}{\tilde{q}_{\alpha/2}}\right) - E\left(\frac{\sum_{i=1}^{r-1} c_i Z_i}{\tilde{q}_{1-\alpha/2}}\right) = \frac{\sum_{i=1}^{r-1} c_i \nu_i}{\tilde{q}_{\alpha/2}} - \frac{\sum_{i=1}^{r-1} c_i \nu_i}{\tilde{q}_{1-\alpha/2}}. \quad (6)$$

Comparing (5) with (6) it is clear that normed by the true σ_1^2, the mean lengths of GCIs constructed by R_1 with $c_i = d_i,\ i = 1, ..., r-1$ and by R_2 with $c_i = d_i/\lambda_i,\ i = 1, ..., r-1$ tend to the same quantity as $\sigma_1^2/\sigma^2 \to \infty$.

References

[1] Hannig J., Iyer H., Patterson P.: Fiducial generalized confidence intervals. JASA, 2006, 101: 254–269.

[2] El-Bassiouni M.Y.: Short confidence intervals for variance components. Commun.Statist.-Theory and Methods, 1994, 23(7): 1915–1933.

[3] Park D.J., Burdick R.K.: Performance of confidence intervals in regression models with unbalanced one-fold nested error structure. Commun.Statist.-Simulation and Computation, 2003, 32: 717–732.

[4] Lidong E., Hannig J., Iyer H.: Fiducial intervals for variance components in an unbalanced two-component normal mixed linear model, preprint.

[5] Li X., Li G.: Comparison of confidence intervals on the among group variance in the unbalanced variance component model. Journal of Statistical Computation and Simulation, 2007, 77: 477–486.

[6] Arendacká B.: A modification of the Hartung-Knapp confidence interval on the variance component in two-variance-component models. Kybernetika, 2007, 43(4): 471–480.

[7] Arendacká B.: Approximate and generalized approaches to confidence intervals on a variance component. To appear in Tatra Mountains Mathematical Publ..

[8] Arendacká B.: Generalized confidence intervals on the variance component in mixed linear models with two variance components. Statistics, 2005, 39: 275–286.

[9] Olsen A., Seely J., Birkes D.: Invariant quadratic unbiased estimation for two variance components.The Annals of Statistics, 1976, 4 (5): 878–890.

[10] Williams J.S.: A confidence interval for variance components. Biometrika, 1962, 49: 278–281

[11] Tukey J.W.: Components in regression. Biometrics, 1951, 7: 33,-69.

[12] Tate R.F., Klett G.W.: Optimal confidence intervals for the variance of a normal distribution. JASA, 1959, 287: 674–682.

[13] Michalski A., Zmyslony R.: Testing hypotheses for variance components in mixed linear models. Statistics, 1996, 27: 297–310.