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A Frequency-Time Algorithm of Parameter Estimation for Sinusoidal Signal in Noise

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In this paper, a computationally efficient and high precision parameter estimation algorithm with frequency-time combination is proposed to improve the estimation performance for sinusoidal signal in noise, which takes the advantages of frequency- and time-domain algorithms. The noise influence is suppressed through spectrum analysis to get coarse frequency, and the fine frequency is obtained by de-noising filtering and using linear prediction property. Then, estimation values of the amplitude and initial phase are obtained. The numerical results indicate that the proposed algorithm makes up for the shortcomings of frequency- and time-domain algorithms and improves the anti-interference performance and parameter estimation accuracy for sinusoidal signal.

Keywords: Frequency-time combination, parameter estimation, spectrum analysis, de-noising, linear prediction property.

1. INTRODUCTION

Accurate parameter estimation for real-value sinusoidal signal in additive white Gaussian noise is a fundamental problem in a wide range of fields, and the studies of parameter estimation algorithms have an important theoretical significance and application value [1], [2]. The samples of a single-tone sinusoidal signal can be described as:

$$x_n = s_n + z_n \qquad n = 0,1,\cdots,N-1 \tag{1}$$

where $s_n = A\cos(\omega n + \theta)$, $A > 0$, $\omega \in (0,\pi)$, $\theta \in (-\pi,\pi)$ are unknown but deterministic constants that represent the amplitude, frequency, and initial phase of the sampled signal, respectively. The subscript n denotes index number, and N is the signal length. z_n is zero-mean, additive white Gaussian noise with variance σ^2 . The signal to noise ratio (SNR) is defined as $SNR = 10\lg\left(A^2/\left(2\sigma^2\right)\right)$, and the unit of SNR is dB.

The frequency ω is the most crucial parameter of the sampled signal. As long as the frequency is estimated, the estimation of amplitude and initial phase can be obtained subsequently. Therefore, the frequency estimation algorithms are mainly introduced in this paper [3].

In the last decades, many frequency estimation algorithms for sinusoidal signal in noise have been proposed, which can be classified into two main categories: the frequency- and time-domain algorithms [4].

The frequency-domain algorithms are mainly based on the Discrete Fourier Transform (DFT) [5]. It is well known that the real-value sinusoidal signal has positive- and negative-frequency components, which superpose and interact with each other. Therefore, the results of frequency-domain algorithms are vulnerable to the spectrum leakage caused by the negative-frequency component. In some researches, windowing approach is adopted to reduce the influence of spectrum leakage [6], which improves the frequency estimation accuracy, but there is an obvious estimation bias in windowing algorithms. Reference [7] proposed a new frequency estimation algorithm, which can improve the estimation performance by filtering out the negative-frequency component. However, when the signal frequency is close to 0 or SNR is in a medium or high state, the estimation accuracy will decrease. Reference [8] estimated signal frequency by incorporating an iterative leakage subtraction strategy, which enhances estimation performance at high SNR, and the algorithm called Ye that has the best estimation performance in frequency-domain algorithms. Nevertheless, when the signal frequency is low, the estimation accuracy decreases with the increase of SNR.

The time-domain algorithms can be classified as linear prediction algorithms [9], autocorrelation algorithms [10], and other algorithms with more computational complexity, such as MUSIC and ESPRIT algorithms [11]. The linear prediction property of the sinusoids is usually used to estimate signal frequency, which is a common frequency estimation algorithm, but it is easily affected by noise [12]. Reference [13] presented an autocorrelation estimation algorithm to estimate frequency by calculating the autocorrelation function of the sampled signal. Due to the

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influence of non-integer period sampling, the estimation accuracy is characterized by periodic deterioration. In order to overcome this drawback, the autocorrelation function is redefined in [14], where the algorithm is called PCA, which has the best frequency estimation accuracy in time-domain algorithms. However, the PCA has a high computation complexity, which reduces the real-time performance.

To sum up, the frequency-domain algorithms have the advantages of better anti-interference properties and real-time performance. However, the estimation accuracy decreases obviously because of the spectrum leakage influence under the condition of high SNRs or low frequencies. The time-domain algorithms are easily affected by noise and non-integer period sampling, but the estimation accuracy is better at high SNRs. In order to enhance the universality and improve the estimation performance for sinusoidal signal in noise, a novel parameter estimation algorithm is proposed in this paper, which combines the advantages of frequency- and time-domain algorithms and overcomes their shortcomings.

The rest of the paper is organized as follows: In Sec. 2, we give the detailed estimation procedure. In Sec. 3, we analyze the computational complexity of the proposed algorithm and compare it with other excellent algorithms. In Sec. 4, the estimation performance of the proposed algorithm is verified by numerical simulations. Finally, the full text is summarized in Sec. 5.

2. NEW ESTIMATION ALGORITHM

The proposed estimator is realized in two steps: coarse estimation and fine estimation. In the first stage, to suppress the noise influence, the sampled signal is processed by spectrum analysis, and the coarse frequency is pre-estimated. In the second stage, the noise is further mitigated through a de-noising filter, and the fine frequency is estimated by linear prediction property. Then, the estimation values of amplitude and initial phase are obtained. It is worth noting that the fine estimation procedure of frequency was proposed in [15], which was called STMB and was designed for damped sinusoidal signal with single-tone in noise. When STMB is directly used to process the signal in this paper, the parameter accuracy is poor under low SNR condition. The parameter accuracy can be improved by adding iterations, but the computation complexity will increase significantly. The proposed algorithm uses the idea of STMB to improve the parameter estimation accuracy, and the results compared with STMB will be discussed in the next section.

A. Coarse estimation

To improve the calculation resolution, we append N zero points after the sampled signal x_n, and the zero-padding signal is $x_{2N} = [x_0, x_1, \ldots, x_{N-1}, 0, 0, \ldots, 0]$. Then, the x_{2N} is processed by 2N-point fast Fourier transform (FFT), and the index number k_0 is obtained.

$$Y_0(k) = DFT\left(x_{2N}\right) \quad k = 1, 2, \ldots, N-1 \tag{2}$$

$$k_0 = \arg\max \left|Y_0(k)\right|^2 \tag{3}$$

where argmax($\bullet$) denotes the argument of the maximum value of function $\bullet$.

Then, the coarse frequency $\hat{\omega}_0$ is calculated by:

$$\hat{\omega}_0 = k_0 \frac{\pi}{N} \tag{4}$$

Where $\hat{\bullet}$ is the estimation value of $\bullet$.

After adding zero signal to sampled signal, the coarse frequency estimation becomes more accurate. However, the computation cost increases. To solve this drawback, a fast procedure to get k_0 is designed. Firstly, the sampled signal is processed by N-point FFT directly, and the index number is assumed as k_1.

$$Y_1(k) = DFT\left(x_n\right) \quad k = 1, 2, \ldots, N/2-1 \tag{5}$$

$$k_1 = \arg\max \left|Y_1(k)\right|^2 \tag{6}$$

There is no doubt that k_0 equals one of $2(k_1 - 0.5)$, $2k_1$ and $2(k_1 + 0.5)$.

Then, the index number k_2 of the highest magnitude of three points spectrum, $Y_2(k_1 - 0.5)$, $Y_2(k_1)$, and $Y_2(k_1 + 0.5)$, can be obtained.

Finally, the k_0 is computed by $k_0 = 2k_2$.

In addition, the computational complexity of two procedures will be analyzed in the next section.

B. Fine estimation

According to the linear prediction property of sinusoidal signal, the noiseless signal s_n can be expressed as:

$$s_n = 2\cos(\omega) s_{n-1} - s_{n-2} \tag{7}$$

and the prediction coefficient a is:

$$a = 2\cos(\omega) \tag{8}$$

When the coarse estimation value of frequency is acquired, the initial value of a is obtained.

To decrease the noise influence, a de-noising filter is designed in [15], and the transfer function is:

$$H(z) = \frac{1}{1 - az^{-1} + z^{-2}} \tag{9}$$

According to the z transform on both sides of (7),

$$\left(1 - az^{-1} + z^{-2}\right) S(z) = 0 \tag{10}$$

where $S(z) = \frac{b_0 + b_1 z^{-1}}{1 - az^{-1} + z^{-2}}$, $b_0 = A\cos\theta$, $b_1 = -A\cos(\omega - \theta)$.

The sampled signal x_n can be expressed as:

$$x_n + x_{n-2} = a x_{n-1} + b_0 \delta_n + b_1 \delta_{n-1} + z_n \tag{11}$$

where δ_n denotes the unit impulse signal.

Computing N samples of the filter's impulse response $u_n = H\{\delta_n\}$ and de-noising the sampled signal $v_n = H\{x_n\}$, the filtered signals v_n and u_n fulfill the relationship:

$$v_n + v_{n-2} = a v_{n-1} + b_0 u_n + b_1 u_{n-1} + z_n \tag{12}$$

For an N-point sampled signal, the prediction matrix can be constructed by linear prediction property.

$$\begin{bmatrix} v_0 + 0 \\ v_1 + 0 \\ \vdots \\ v_{N-1} + v_{N-3} \end{bmatrix} = \begin{bmatrix} 0 & u_0 & 0 \\ v_0 & u_1 & u_0 \\ \vdots & \vdots & \vdots \\ v_{N-2} & u_{N-1} & u_{N-2} \end{bmatrix} \begin{bmatrix} a \\ b_0 \\ b_1 \end{bmatrix} + \begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{N-1} \end{bmatrix} \tag{13}$$

After filtering, the noise values become small, i.e. $z_n \approx \mathbf{0}$. Then, the prediction coefficients can be calculated by the least squares method, and the fine frequency is obtained according to (8).

$$\hat{\omega} = \cos^{-1}(a/2) \tag{14}$$

According to the estimation value of frequency, the complex amplitude can be calculated by an iterative procedure to suppress the spectrum leakage influence.

$$c = \frac{1}{N} \sum_{n=0}^{N-1} \left(x_n - c^* e^{-i\hat{\omega} n} \right) e^{-i\hat{\omega} n} \tag{15}$$

where $c = \frac{A}{2} e^{i\theta}$, $c^* = \frac{A}{2} e^{-i\theta}$, and the initial value of c^* is 0.

Thus, the estimation values of amplitude and initial phase are obtained.

$$\begin{cases} \hat{\theta} = \angle c \\ \hat{A} = 2|c| \end{cases} \tag{16}$$

where $|\bullet|$ and $\angle\bullet$ denote the modulus and angle of complex $\bullet$, respectively.

The detailed procedure of the proposed algorithm can be summarized as follows:

1. The coarse frequency is calculated by (4), to obtain the initial prediction coefficient.
2. The precision prediction coefficient is calculated by (13).
3. The fine frequency is estimated by (14), and the fine initial phase and amplitude are obtained subsequently by (15) and (16).

3. Computation complexity analysis

In this section, the computational complexity of frequency estimation is analyzed, and the comparison results with other excellent algorithms are presented in Table 1. and Fig.1.

In the analysis, we neglect the operations where the complexity is O(1). Firstly, we analyze the computation cost of 2N-point FFT procedure and N-point FFT procedure for finding index number k_0. For 2N-point FFT, the addition is $6N\log_2^{2N}$ and multiplication is $4N\log_2^{2N}$. Whereas, for N-point FFT procedure, the addition is $3N\log_2^{N}+4N$ and the multiplication is $2N\log_2^{N}+6N$. Hence, the N-point FFT procedure reduces the computational cost significantly. The PCA needs more additions and multiplications than other algorithms, because autocorrelation functions are calculated several times in the frequency coarse estimation stage. In addition, the PCA needs to construct the reference signal, which will calculate 2N sine/cosine functions. The Ye and STMB belong to iterative algorithms, and the iteration times of Ye and STMB are fixed to 3 and 2 for convergence according to the references, respectively. The Ye uses the FFT algorithm and calculates some iterative interpolation procedures, so the additions and multiplications are smaller than those of the PCA. However, the Ye needs to calculate complex-value exponential that requires two sine/cosine calculations for one operation, which increases the complexity. Because STMB needs to calculate the prediction matrix, the additions and multiplications are slightly higher than those of Ye, and are also smaller than those of PCA. Although the proposed algorithm uses the idea of STMB, our approach does not need iterative calculation. The computational cost of the new algorithm is slightly bigger than that of Ye, but is smaller than those of PCA and STMB. Therefore, the proposed algorithm is computationally efficient.

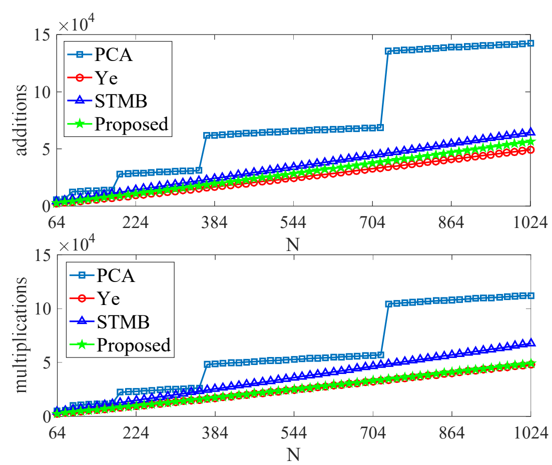


Fig.1. Comparisons of the computational complexity. a) and b) show the comparisons of additions and multiplications computations, respectively.

Table 1. Comparison of the computation complexity.

algorithm	additions	multiplications	Sine/cos
PCA[15]	$3\times 2^{M}\left[\log_2^{2N}\right]+2^{M}+3N$	$2^{M+1}[\log_2^{2N}]+2^{M+1}+41N/3$	N
Ye[7]	$3N\log_2^N+18N$	$2N\log_2^N+27N$	18N
STMB	$63N$	$66N$	6
Proposed	$3N\log_2^N+25N$	$2N\log_2^N+28N$	4N

where $M=[\log_2^{2N}]+1$.

4. NUMERICAL RESULTS

To verify the estimation performance of the proposed algorithm, the simulations are carried out under different conditions. The results are compared with those of PCA, Ye, and STMB. Meanwhile, the Cramer-Rao lower bound (CRLB) is used as a test standard [16], and the CRLB of the frequency, initial phase, and amplitude estimation values of real-value sinusoidal signal are:

$$\begin{cases} \operatorname{var}(\hat{\omega}) \geq \dfrac{12}{N(N^2-1)SNR} \\ \operatorname{var}(\hat{\theta}) \geq \dfrac{2(2N-1)}{N(N+1)SNR} \\ \operatorname{var}(\hat{A}) \geq \dfrac{A^2}{NSNR} \end{cases} \tag{17}$$

In these tests, we fix $A=1$, $N=128$, and θ is random from $-\pi$ to π. The mean square errors (MSEs) of the estimated parameter value are calculated over 5000 Monte Carlo simulations.

Variable SNRs: we first examine the frequency estimation performance versus SNRs, varying from -10 to 50 dB, in steps of 2 dB. In addition, $k_0=2$, and δ is random from -0.5 to 0.5. The numerical results are shown in Fig.2.

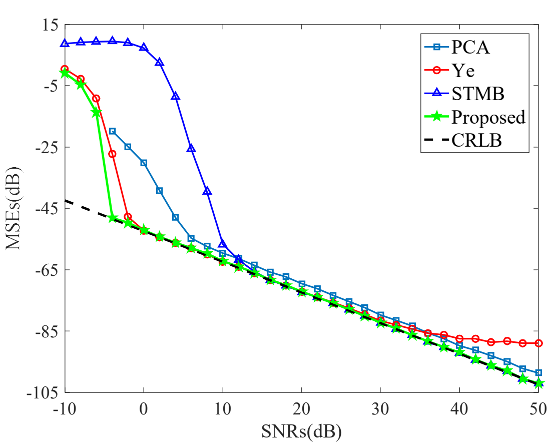


Fig.2. MSEs of $\hat{\omega}$ versus *SNRs*.

Our approach outperforms the other algorithms, and the MSEs are extremely close to the CRLB when SNRs > -4 dB. The estimation results of STMB designed for damped sinusoidal signal are close to CRLB in middle or high SNRs, but the anti-interference performance is very poor when SNRs < 15 dB. The Ye performs well at low SNRs, but it is characterized by MSEs saturation when SNRs > 30 dB. The PCA algorithm suffers from the interference, and there is an estimation deviation with CRLB about 3 dB.

Variable frequencies: Then, we verify the frequency estimation performance versus frequencies, varying from 0.025π to 0.5π, in steps of 0.025π, and $SNR=10dB$. The numerical results are shown in Fig.3.

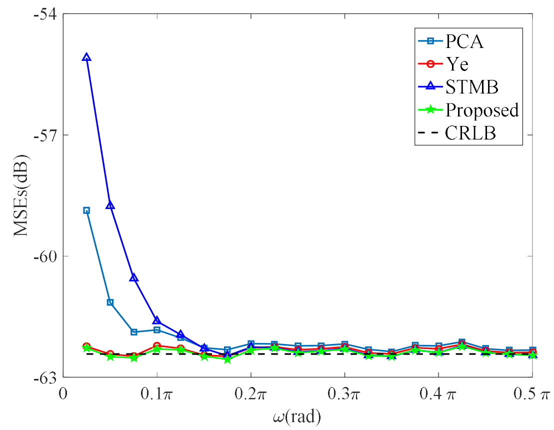
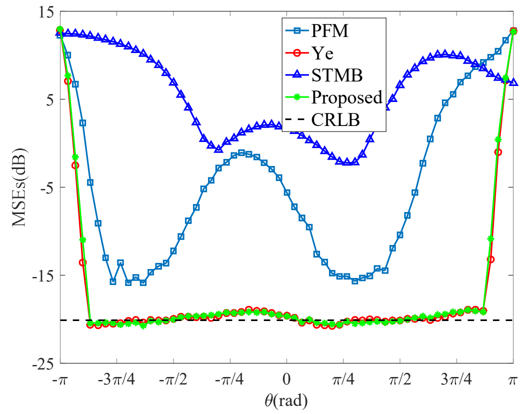


Fig.3. MSEs of $\hat{\omega}$ versus ω.

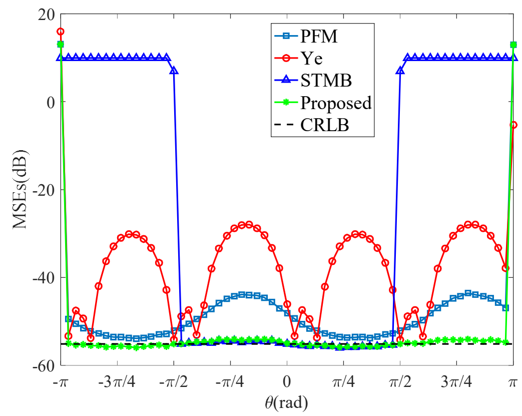
Our approach has considerable estimation accuracy, and is better than other ones for all frequencies. The estimation performance of STMB is not good at low frequencies, and the estimation effect meliorates with the increase of frequencies. The PCA also has a low estimation accuracy at low frequencies when $\omega \leq 0.1\pi$, and there is an estimation deviation at medium and high frequencies. In addition, Ye has a good estimation performance, but it is a little worse than our approach.

Variable initial phase: Then, we vary the initial phase to examine estimation performance of the initial phase, ranging from $-\pi$ to π, in steps of $\pi/30$, $k_0=2$, and $\delta=0.2$. In addition, SNR = 5 dB or SNR = 40 dB, and the numerical results are shown in Fig.4., respectively.

In high SNRs condition, the proposed algorithm has a better initial phase estimation performance than those of other algorithms. The estimable range of initial phase is reduced slightly with the decrease of SNRs, but the estimation results are closer to CRLB than to other ones. The phase estimation range of STMB is restricted in high SNRs, and the estimation performance sharply declines in low SNRs. The PFM proposed in [13] is an excellent time-domain algorithm for phase estimation, but estimation effect is not good when the signal frequency is small. The frequency-domain algorithm, Ye, is close to CRLB when SNR is low because of the good anti-interference performance, but periodicity in the MSEs behavior is evident at high SNRs.



a) SNR = 5 dB



b) SNR = 40 dB

Fig.4. MSEs of $\hat{\varphi}$ versus θ when SNR = 5 dB and SNR = 40 dB.

Variable SNRs: Finally, let us consider variable SNRs to verify the amplitude estimation performance, ranging from -10 to 50 dB, in steps of 2. In addition, $k_0=2$, $\delta=0.2$, and θ is random from $-\pi$ to π. The test results are shown in Fig.5.

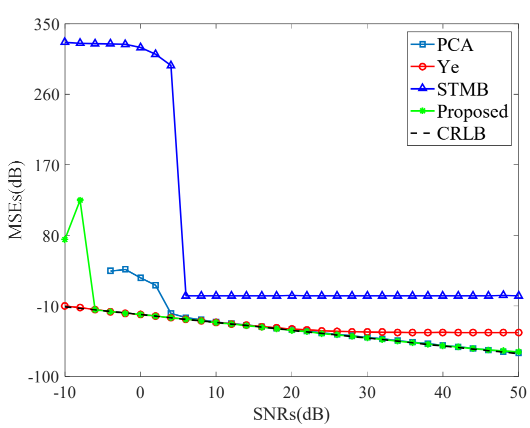


Fig.5. MSEs of $\hat{A}$ versus. $SNRs$.

In low SNRs, our approach has a little estimation gap with Ye, but our approach performs better than other ones as SNR increases. Meanwhile, Ye has less effective estimation results because of spectrum leakage in high SNRs. The estimation results of PCA and STMB are affected by noise, and STMB has an obvious estimation deviation with CRLB.

5. Conclusion

In this paper, a novel parameter estimation algorithm is proposed to reduce the noise influence and to improve the estimation performance. The proposed algorithm makes full use of the advantages of frequency- and time-domain algorithms. The noise influence is suppressed by FFT of frequency-domain algorithms and de-noising filtering of time-domain algorithms and the estimation performance is further improved by linear prediction property. Although the algorithm is simple, the idea is novel.

The numerical results indicate that the frequency estimation performance of the proposed algorithm is superior to other excellent frequency- and time-domain algorithms under different conditions. The anti-interference and real-time performance are enhanced, and the MSEs of the frequency estimation values achieve the CRLB. Especially, when the signal frequency is low and the SNRs are high, the estimation advantage becomes more obvious. Meanwhile, the estimation accuracy of initial phase and amplitude are better than the other ones.

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