Interdependence Measure Based on Correlation Dimension

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Abstract. *We use correlation dimension to study unidirectional coupling and synchronization of interconnected dynamical systems. The method is illustrated on coupled Hénon systems with variable coupling strengths. Results show that correlation dimension can be used to estimate the active degrees of freedom of the coupled dynamical systems, detect full synchronization, and reveal the direction of coupling.*

Keywords: Correlation Dimension, Hénon, Synchronization, Unidirectional Coupling

1. Introduction

Nowadays, the study of synchronization and drive-response relationships between oscillators or between dynamical systems in general is a topic of increasing interest. Applications are found, among others, in domains as economy, chemistry, climatology, causal relations in electrical activity of brain, or cardio-respiratory relations.

In this paper the following definition of *generalized synchronization* will be used:

Denote by X and Y two dynamical systems. Assume they are unidirectionally coupled:

$$\dot{x}(t) = F(x(t)),$$

$$\dot{y}(t) = G(y(t), x(t)),$$

where the x and y are the state vectors of driving system X and the driven response Y.

If the following relation

$$y(t) = \Psi(x(t))$$

exists for some smooth and invertible function Ψ, then there is said to be a *generalized synchronization* between X and Y. If Ψ is an identity, the synchronization is called identical. After some definitions, Ψ does not need to be smooth. E.g., Pyragas defines as *strong* and *weak synchronizations* the cases of smooth and non-smooth transformations [1].

Detection of Synchronization

Identical synchronization of systems of known dynamics can easily be checked visually. Run the response system from two different initial conditions. If the trajectories get synchronized after some transient, they are obviously independent of the initial conditions, but dependent on the driver.

In case of generalized synchronization of X and Y any recurrence of X implies a recurrence of Y. In real data, exact recurrences cannot be expected. However, we can assume that closeness of points in the state space X implies a closeness of the contemporary states of Y (based on the assumption of the existence of a smooth map between the two trajectories). It can be tested by making nonlinear forecasts of x_i using local neighborhood and comparing the quality of the forecasts with that of forecasts based on the equal time partners of the nearest neighbors of y_i.

Detection of Coupling without Synchronization

The direction of coupling can only be uncovered when the coupling is weaker than the threshold for emergence of synchronization. Once the systems are synchronized, there is a one-to-one relation between the states of the systems. Then the future states of the driver X can be predicted from the response Y equally well as vice versa.

First mathematical approaches to the coupling detection include the notion of Granger causality which evaluates the causal relations of time series by study of predictability in autoregressive models [2]. However, the linear concept requires generalizations to enable investigation of complex nonlinear processes. Therefore, new approaches were proposed, including nonlinear Granger causality, transfer entropy, cross predictabilities, measure based on conditional mutual information, measures evaluating distances of conditioned neighbors in reconstructed state spaces, etc. (see [3, 6] and references therein).

In this study, new approach for detecting driver – response relationships is introduced. The method utilizes the fact that the unsynchronized coupling of the systems is more complex than the driving system alone.

As a testing example, we use unidirectionally coupled identical Hénon maps at different coupling strengths.

2. Subject and Methods

To detect causal relationships we propose to use the well-known complexity measure called correlation dimension (D_2), computed after Grassberger-Proccacia algorithm [4].

Suppose we have a driving system X and response Y with a unidirectional coupling. Let us create $X+Y$ combining state vectors of X and Y.

Then our approach to study the coupling effects is based on the next expectations:

- for uncoupled X and Y the active degrees of freedom (estimated through the correlation dimension) of the combined system are equal to the sum of the active degrees of freedom of X and Y
- for coupled but not synchronized case, the number of active degrees of freedom of $X+Y$ is equal to that of the response Y and higher than the number of degrees of freedom of the driver
- the dimension of the attractor of response system saturates to the dimension of the driving system's attractor as the coupling reaches the synchronization level ($D_X=D_Y=D_{X+Y}$)

Similar approach was proposed in [5], leading to the index $(D_X+D_Y)/D_{X+Y}$ named dynamic complexity coherence measure. However, the index revealed the presence of the coupling but not the direction of the coupling. Regarding the later task, the authors only suggest to utilize the variability in the correlation dimension, which is supposed to be greater in the case of the response system Y (influenced by dynamics of X) in comparison to the dimension of the driving system X alone.

Unlike in [5], we are going to define the following *dimension-based interdependence measures*:

$$D(X|Y) = D_Y/D_{X+Y}, \quad D(Y|X) = D_X/D_{X+Y}, \quad \text{and their difference} \quad \Delta D = D(X|Y) - D(Y|X),$$

where D_X is the correlation dimension of the driving dynamics X (estimated in state space of dimension m_X), D_Y is the correlation dimension of response Y (in state space of dimension m_Y)

and D_{X+Y} is the correlation dimension of the combined system estimated in state space of dimension m_X+m_Y.

Our measure ΔD should be zero when the systems are independent or synchronized, positive when X is driving Y, and negative when Y is the driver and X is the response system.

3. Results and Discussion

To demonstrate the effectiveness of the proposed technique we will concentrate on an example of unidirectionally coupled identical Hénon maps with the driving system

$$x_1[n+1] = 1.4 - x_1^2[n] + 0.3x_2[n]$$

$$x_2[n+1] = x_1[n]$$

and the response system

$$y_1[n+1] = 1.4 - (Cx_1[n]y_1[n] + (1-C)y_1^2[n]) + 0.3y_2[n]$$

$$y_2[n+1] = y_1[n] \,.$$

The same case of unidirectionally coupled system was also studied in [3, 5, 6]. Starting from any initial point, after a transient regime the dynamics of Hénon map is trapped to a chaotic attractor of correlation dimension of about 1.22 (see Fig. 1 on the left).

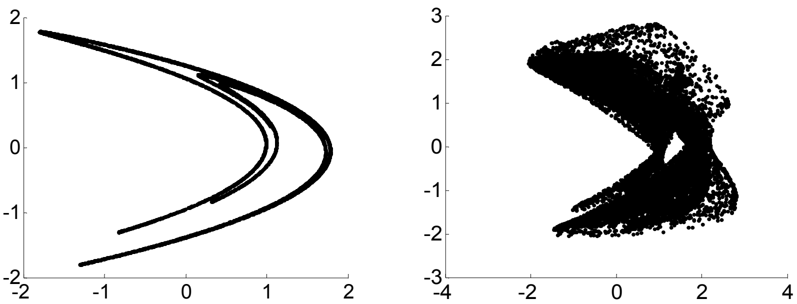

Fig. 1. Driving Hénon attractor (left) and the response system for coupling $C = 0.6$ (right).

In our toy example the coupling strength C varies from 0 to 0.8. The attractor of the response looks the same as the driver for $C = 0$ (independent systems) and $C > 0.72$ (identical synchronization). An example of unsynchronized coupling ($C = 0.6$) can be seen in Fig.1.

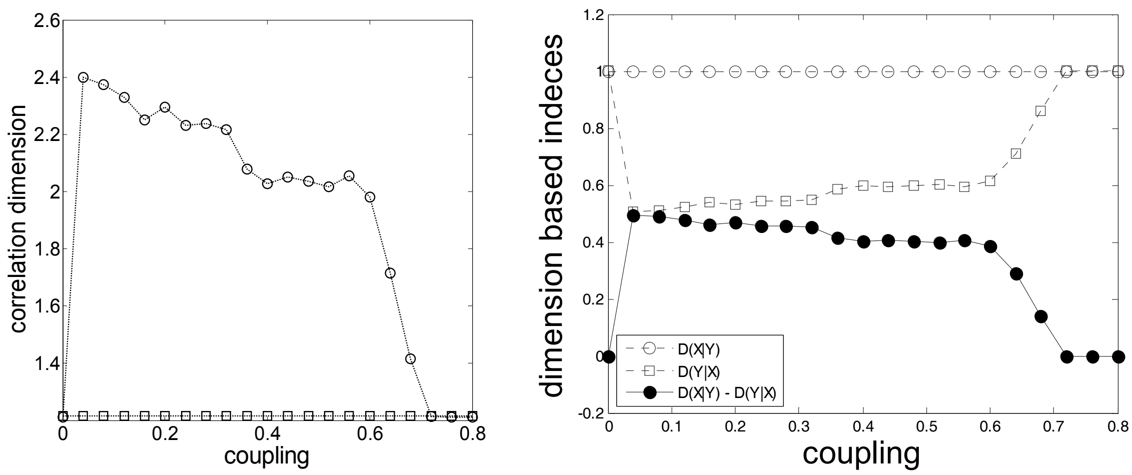

Fig. 2. Left: Estimates of D_2 for driving Hénon system (squares), and the coupled system (circles). Right: The dimension-based interdependence measures for coupled Hénon system.

Our results show that, in case of noise-free and long data, detection of the driver and response in the coupled systems by correlation dimension is possible.

Estimates of correlation dimension of the Hénon map, computed for 200000 numerically generated data (Fig. 2, on the left) lead to values about $D_X = 1.22$ for the driving system and values D_Y below 2.44 for the coupled system. Then the dimension-based index ΔD (on the right on Fig. 2), indicates by positive values that X is the driver and Y the response system and also clearly reveals the onset of synchronization by drop to zero for the synchronization threshold.

4. Conclusions

The reliable detection of the coupling from experimental signals is an important task. The dimension-based approach introduced in this paper enables very reliable detection of the direction of coupling and the threshold of synchronization for artificial benchmark data. It was demonstrated on an example of coupled Hénon maps, which allows the use of a large amount of data to estimate the probability distributions in state spaces of considerable dimensions. However, the world of real-life experiments is much more complicated: the time series are noisy and of finite length and the representative state portrait in multidimensional space is usually reconstructed from single measured observable. In such cases, the effectiveness of our dimension-based measures will depend on the achievable quality of the dynamics reconstruction.

At this point, the most important contribution of the proposed measure is that it can be used as a standard for preliminary testing of other interdependence measures that are continuously invented to evaluate synchronization phenomena in real coupled systems.

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