New Procedure for Calculating the Uncertainty of One Output Quantity in Calibration Certificates

[1]G. Wimmer, [2]V. Witkovský

[1]Mathematical Institute Slovak Academy of Sciences, Bratislava, Slovakia,
Faculty of Natural Sciences, Matej Bel University, Banská Bystrica, Slovakia, and
Department of Mathematics and Statistics, Faculty of Science, Masaryk University,
Brno, Czech Republic,
[2]Institute of Measurement Science, Slovak Academy of Sciences, Bratislava, Slovakia
Email: wimmer@mat.savba.sk

Abstract. *Using the software "Tail probability calculator" (see [1],[2]), the (1-α)-confidence region for unknown output quantity in calibration certificates is stipulated. Specified is also the real coverage probability corresponding to the expanded uncertainty of the measurand. This expanded uncertainty is determined according to regulation [3].*

Keywords: One Output Quantity Calibration Certificate, Confidence Interval, Tail Probability Calculator

1. Introduction

This contribution is based on EA (European co-operation for Accreditation) document EA 4/02. For calibration laboratories is there described "an unambiguous and harmonised way of evaluating and stating the uncertainty of measurement" (see [3], p.4). In accordance with EAL decision (see [3]) calibration laboratories shall state an expanded uncertainty U of measurement, obtained by multiplying the standard uncertainty of the output estimate by coverage factor k. For k=2 the assigned expanded uncertainty (in the case where a normal (Gaussian) distribution can be attributed to the measurand) corresponds to a coverage probability of approximately 95%. On p. 12 in [3] is claimed, that this condition is fulfilled in the majority of cases encountered in calibration work. But in effect in (almost) all real cases it is not possible to attribute a normal distribution to the measurand (see all examples in [3]). This contribution answers the questions:

(i) How to obtain the *(1-α)*-confidence region for unknown output quantity in calibration certificates ?

(ii) What is the real coverage probability corresponding to expanded uncertainty determined according to regulation [3] ?

2. Subject and Methods

As stated in [3], p. 5, the set of input quantities $X_1,X_2,\dots,X_n,Z_1,\dots,Z_m$ can be grouped into two categories according to the way in which the value of the quantity have been determined.

(a) quantities $X_1,\dots,X_n$, whose estimate are directly determined in the current measurement;

(b) quantities $Z_1,\dots,Z_m$, whose estimate are brought into the measurement from external sources.

Considered case of calibration deals with only one measurand or output quantity Y.

The **theoretical** (mathematical, physical) model of calibration is understood as the equation

$$Y = f(X_1,\dots, X_n, Z_1,\dots, Z_m), \tag{1}$$

where the model function f represents the procedure of the measurement and the method of evaluation. Measuring the real (true) value X we mathematically denote by the random variable ξ_X and the estimate of the quantity X (the measured value) is denoted as x.

The **stochastic** model of calibration is understood as the equation

$$\xi_Y = f(\xi_{X_1},\dots,\xi_{X_n},\xi_{Z_1},\dots,\xi_{Z_m}). \tag{2}$$

In accordance with [EA4/02] we suppose that

-- measurements $\xi_{X_1},\dots,\xi_{X_n}$, $\xi_{Z_1},\dots,$ ξ_{Z_m} are independent,

-- probability distributions of measurements $\xi_{X_1},\dots,\xi_{X_n}$, $\xi_{Z_1},\dots,$ ξ_{Z_m} are known, measurements obey one of the following distributions: normal, rectangular, triangular, Student's t,

-- estimates of all quantities $X_1,X_2,\dots,X_n,Z_1,\dots,Z_m$ i.e. the realizations $x_1,\dots,x_n,z_1,\dots z_m$ of random variables $\xi_{X_1},\dots,\xi_{X_n}$, $\xi_{Z_1},\dots,$ ξ_{Z_m} are known.

Saying more precise, if the mean value of the random variable ξ_j is $\mathrm{E}(\xi_j)$ (unknown), then the distribution of ξ_j - $\mathrm{E}(\xi_j)$ is normal with zero mean and known variance or rectangular with zero mean and known support or triangular with zero mean and known support or Student's t with known degrees of freedom.

The stochastic model of calibration (2) we shall linearize in realizations $x_1,\dots,x_n,z_1,\dots z_m$ (using Taylor series expansion) and neglect the terms of second and higher order. So we obtain

$$\xi_Y \cong f(x_1,\dots,x_n,z_1,\dots,z_m) + \left.\frac{\partial f(\xi_{X_1},\dots,\xi_{X_n},\xi_{Z_1},\dots,\xi_{Z_m})}{\partial \xi_{X_1}}\right|_{\xi_{X_1}=x_1,\dots,\xi_{Z_m}=z_m} (\xi_{X_1} - x_1) + \dots$$

$$+ \left.\frac{\partial f(\xi_{X_1},\dots,\xi_{X_n},\xi_{Z_1},\dots,\xi_{Z_m})}{\partial \xi_{X_n}}\right|_{\xi_{X_1}=x_1,\dots,\xi_{Z_m}=z_m} (\xi_{X_n} - x_n) +$$

$$+ \left.\frac{\partial f(\xi_{X_1},\dots,\xi_{X_n},\xi_{Z_1},\dots,\xi_{Z_m})}{\partial \xi_{Z_1}}\right|_{\xi_{X_1}=x_1,\dots,\xi_{Z_m}=z_m} (\xi_{Z_1} - z_1) + \dots$$

$$+ \left.\frac{\partial f(\xi_{X_1},\dots,\xi_{X_n},\xi_{Z_1},\dots,\xi_{Z_m})}{\partial \xi_{Z_m}}\right|_{\xi_{X_1}=x_1,\dots,\xi_{Z_m}=z_m} (\xi_{Z_m} - z_m) =$$

$$= (a_0 - a_1x_1 - \dots - a_nx_n - b_1z_1 - \dots - b_mz_m) + a_1\xi_{X_1} + \dots + a_n\xi_{X_n} + b_1\xi_{Z_1} + \dots + b_m\xi_{Z_m} =$$

$$= A + a_1(\xi_{X_1} - \mathrm{E}(\xi_{X_1})) + \dots + a_n(\xi_{X_n} - \mathrm{E}(\xi_{X_n})) + b_1(\xi_{Z_1} - \mathrm{E}(\xi_{Z_1})) + \dots + b_m(\xi_{Z_m} - \mathrm{E}(\xi_{Z_m})) =$$

$$= A + \xi \ ,$$

where A is unknown, $a_1,\dots,a_n,b_1,\dots,b_m$ are known values. Understandably A is considered as the true value of the output quantity Y. That is $\xi_Y = Y + \xi$. The distribution of ξ is a linear combination (with known coefficients) of known distributions, each of them has zero mean. In addition the distribution of ξ is according to our assumptions symmetrical around zero. Using the software [1] or [2] we can easy calculate the *(1-α/2) quantile* $\gamma_{(1-\alpha/2)}$ of ξ and obtain the interval $(-\gamma_{(1-\alpha/2)}, \gamma_{(1-\alpha/2)})$ for which is

$$P\{-\gamma_{(1-\alpha/2)} \le \xi \le \gamma_{(1-\alpha/2)}\} = 1-\alpha \ .$$

It holds

$$P\{Y-\gamma_{(1-\alpha/2)} \leq Y+\xi = \xi_Y \leq Y+\gamma_{(1-\alpha/2)}\}=1-\alpha$$

i.e.

$$P\{\xi_Y - \gamma_{(1-\alpha/2)} \leq Y \leq \xi_Y + \gamma_{(1-\alpha/2)}\}=1-\alpha$$

and so

$$(\xi_Y - \gamma_{(1-\alpha/2)}, \xi_Y + \gamma_{(1-\alpha/2)}) \tag{3}$$

is the $(1-\alpha)$-confidence interval for measured output quantity Y. If the realization of ξ_Y is $y=f(x_1,\dots,x_n,z_1,\dots,z_m)$, the realization of the $(1-\alpha)$-confidence interval is $(y - \gamma_{(1-\alpha/2)}, y + \gamma_{(1-\alpha/2)})$.

3. Example

We shall demonstrate the above mentioned procedure for calculating the uncertainty of one output quantity in calibration certificates on the example given in Supplement 1 of [3], p. 29 "Calibration of a weight of nominal value 10 kg" (exactly on the little corrected version of this example given in [4], p.5).

The unknown output quantity Y is the unknown conventional mass m_X obtained from (1), where

$X_1 = \delta m$ is the (measured) difference in mass between the unknown mass and the standard; there were realized three measurements $\xi_{\delta m}^{(1)}$, $\xi_{\delta m}^{(2)}$, $\xi_{\delta m}^{(3)}$ using the substitution method and substitution scheme ABBA ABBA ABBA, all measurements are supposed to have normal distribution, the realization of $\xi_{\delta m} = 1/3\ (\xi_{\delta m}^{(1)} + \xi_{\delta m}^{(2)} + \xi_{\delta m}^{(3)})$ is 0,020 g and the estimate of its standard deviation (obtained from prior evaluation) is 14,4 mg;

$Z_1 = m_S$ is the conventional mass standard, the calibration certificate gives the value of 10 000,005 g (the realization of ξ_{mS}) with an associated normal distribution of ξ_{mS} and standard uncertainty of 22,5 mg;

$Z_2 = \delta m_D$ is the drift of value of the standard since its last calibration, the distribution of $\xi_{\delta mD}$ is (assumed to be) rectangular in the interval 0 - 15 mg and its realization (estimate) is set to 0;

$Z_3 = \delta m_C$ is variation due to eccentricity, the distribution of $\xi_{\delta mC}$ is (assumed to be) rectangular in interval ± 10 mg and its realization is (set to) 0;

$Z_4 = \delta B$ is air buoyancy, the distribution of $\xi_{\delta B}$ is (assumed to be) rectangular in interval $\pm 1 \times 10^{-6}$ of the nominal value and its realization is (set to) 0.

The proposed v theoretical model of calibration (1) is

$$m_X = \delta_m + m_S + \delta m_D + \delta m_C + \delta B$$

The expanded uncertainty calculated according to [3] is U=57 mg (coverage factor k=2).

Using the software "Tail probability calculator" (see [1], [2]), the measured mass of the nominal 10 kg weight is 10,000 025 kg and the 0,95-confidence region for this weight is ± 55,4 mg. The 0,9-confidence region for this weight is ± 46,5 mg and we can stipulate using the software [1], [2] any confidence region for the calibrated quantity.

On the other hand, we can determine the unknown coverage probability for the expanded uncertainty calculated according to [3]. In previous example is this coverage probability 0,978.

4. Discussion

The example analysed in this contribution was developed (according to [3], p. 28) by a special expert group. Laboratories should determine the probability distribution of measurements acting in the proposed stochastic model (2) of calibration which is based on suggested theoretical model (1) of calibration. It is recommended in many cases to assess the realizations of the measurements ξ_{Zi} using simulations. Another way for determining the proper confidence interval (3) for calibrated value is via Monte Carlo simulations in accordance with [5].

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