Approximation by Rational Functions.

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The paper deals with analytical functions or experimental measured dependencies approximation by rational functions which is useful in the nonlinear analog function block design, the approximation of inverse functions and calculation of the transfer function parameters from measured data of the amplitude frequency characteristic.

In the technical praxis is very often necessary approximation of dependencies by rational functions. The dependencies are usually given in analytical form or in measured data form.

If the approximated function *f(x)* is given in analytical form for the approximation by rational function we can use the Pade approximation in following form

$$f(x) \cong R_N(x) = \frac{P_r(x)}{Q_s(x)} = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_r x^r}{1 + b_1 x + b_2 x^2 + \cdots + b_s x^s}; \quad N = r + s \tag{1}$$

The most useful of the Pade approximations are those with the degree of numerator equal to, or one more than, the degree of denominator. The coefficient calculation is based on Maclaurin expansion of *f(x)* to make *f(x)* and $R_N(x)$ agree at $x = 0$ and to make the first N derivatives agree at $x = 0$. From these conditions follows next expressions.

$$f(x) \cong c_0 + c_1 x + c_2 x^2 + \cdots c_N x^N = R_N(x); \quad c_i = f^{(i)}(0)/i! \tag{2}$$

$$R_N(x) \cdot Q_s(x) - P_r(x) = 0 \tag{3}$$

From the equations (2), (3) are resultant following matrix expressions

$$\begin{bmatrix} c_r & c_{r-1} \cdots & c_{r-s+1} \\ c_{r+1} & c_r \quad \cdots & c_{r-s+2} \\ \cdots & \cdots & \cdots \\ c_{r+s-2} & c_{r+s-3} & c_{r-1} \\ c_{r+s-1} & c_{r+s-2} & c_r \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_{s-1} \\ b_s \end{bmatrix} = \begin{bmatrix} -c_{r+1} \\ -c_{r+2} \\ \cdots \\ -c_{r+s-1} \\ -c_{r+s} \end{bmatrix}; \quad \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \cdots \\ a_r \end{bmatrix} = \begin{bmatrix} c_0 & 0 & 0 & \cdots 0 \\ c_1 & c_0 & 0 & \cdots 0 \\ c_2 & c_1 & c_0 & \cdots 0 \\ \cdots & \cdots & \cdots & \cdots \\ c_r & c_{r\text{-}1} & c_{r\text{-}2} & \cdots c_{r-s} \end{bmatrix} \begin{bmatrix} 1 \\ b_1 \\ b_2 \\ \cdots \\ b_s \end{bmatrix} \tag{4}$$

From these matrix equations we can calculate the required coefficients $a_0, a_1 \cdots a_r$ and $b_1, b_2 \cdots b_s$ by following matrix expressions

$$[A_2]\cdot[b]=[C]; \qquad [b]=[A_2]^{-1}\cdot[C]; \qquad [a]=[A_1]\cdot[b^*] \tag{5}$$

The dimension of matrices A_2, A_1 are $s \cdot s$ and $(r+1)\cdot(s+1)$. The values r, s in technical applications are usually not larger then 5.

If the function f(x) is given by measured data pair then the rational function approximations can be made by following manner. For every data pair x_i, y_i, i = 0,1...N, is valid expression 1 and we can write following expression

$$y_i = \frac{a_0 + a_1 x_i + a_2 x_i^2 + \cdots + a_r x_i^r}{1 + b_1 x_i + b_2 x_i^2 + \cdots + b_s x_i^s}; \tag{6}$$

where x_i, y_i are known (measured) data and a_j, b_k are required unknown coefficients. The expression (6) we can write in following form too

$$y_i = a_0 + a_1 x_i + a_2 x_i^2 + \cdots + a_r x_i^r - b_1 x_i y_i - b_2 x_i^2 y_i - \cdots - b_s x_i^s y_i \tag{7}$$

By the equation (7) for i = 0,1, ... N we can give following $N+1$ equations written in matrix form

$$[y]=[X]\cdot[a] \tag{8}$$

where

$$[y]=[y_0, y_1, y_2 \ldots y_N]^T; \qquad [a]=[a_0, a_1, a_2, \cdots a_r, b_1, b_2, \cdots b_s]^T$$

$$[X]=\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^r & -y_0 x_0 & -y_0 x_0^2 & \cdots\cdots -y_0 x_0^s \\ 1 & x_1 & x_1^2 & \cdots & x_1^r & -y_1 x_1 & -y_1 x_1^2 & \cdots\cdots -y_1 x_1^s \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_N & x_N^2 & \cdots & x_N^r & -y_n x_N & -y_N x_N^2 & \cdots\cdots -y_N x_N^s \end{bmatrix} \tag{9}$$

If in the system of equations there is N = r + s , then the required coefficients $a_0, a_1 \cdots a_r, b_1, b_2 \cdots b_s$ we can calculate by following expression

$$[a]=[X]^{-1}\cdot[y] \tag{10}$$

If in the system of equation (8) there is $N > r+s$ then we can calculate the required coefficients by following expression (least-square method)

$$[a]=[[X]^T\cdot[X]]^{-1}\cdot[X]^T[y] \tag{11}$$

The approximation by rational function is useful too on calculation of inverse functions, which are necessary in linearization of measurement instrument static characteristic nonlinearitly. The static characteristic of measurement instrument is usually given by measured data pair, x_i , y_i and usually is approximated by polynomial or rational functions in following forms

$$y \cong f(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_N x^N \tag{12}$$

$$y \cong f(x) = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_r x^r}{1 + b_1 x + b_2 x^2 + \cdots + b_s x^s} \tag{13}$$

In the linerarization process is exploited the property of inverse function $\varphi(x)$ which we can express by the following expression

$$\varphi(y) = \varphi(f(x)) = f^{-1}(f(x)) = x \tag{14}$$

and for the rational form of the inverse function it will be valid following record

$$x_i = \frac{A_0 + A_1 y_i + A_2 y_i^2 + \cdots + A_r y_i^r}{1 + B_1 y_i + B_2 y_i^2 + \cdots B_s y_i^s} \tag{15}$$

For $i = 0,1,2\ldots N$ by equation (15) we can write $r + s + 1$ equations for unknown coefficient $A_0 \cdots A_r$, $B_1 \cdots B_s$ and the value of N must be $N \geq r + s$. Then the solution of these equations can be made by the same method which is used in solution of equations (8). The inverse function will have then the following rational form

$$\varphi(x) = \frac{A_0 + A_1 x + A_2 x^2 + \cdots + A_r x^r}{1 + B_1 x + B_2 x^2 + \cdots + B_s x^s} \tag{16}$$

The approximation by rational functions can be used too in approximation of functions sinx, cosx, tgx, arctgx and others in its realization by electric analog function blocks and in calculation of transfer function, parameters from the discrete measured data of the amplitude frequency characteristic. The described methods of the approximations by rational functions enable the calculation of the required parameters by solving linear system of equations and the application of the least-square method in these calculations.

References

[1] Bartkovjak, J. : Linearizácia statických charakteristík snímačov, Jemná mechanika a optika, 1983, č.4, p.99-102.
[2] Bronštejn, I.N.- Semendjajev, K.A.: Príručka matematiky, SVTL Bratislava, 1961.
[3] Gerald, C.F.- Wheatley, P.O.: Applied Numerical Analysis, Addison-Wesley Publishing Company, New York, 1994.