COMPARISON OF UNCERTAINTY ESTIMATES: ALLAN VARIANCE AND SAMPLE VARIANCE

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General scheme for estimation of errors and uncertainties is presented and discussed. Allan variance as an important characteristic of data scatter is compared with classic sample variance. Two approaches are outlined for investigation of Allan variance versus sample variance. The first one is based on the testing of statistical hypotheses, the second one is based on the functional representations of time series.

1. Introduction

Together with tendency to unification of errors and uncertainties estimation, there is also an urgent necessity to extend the set of errors estimates. The latter tendency is clearly confirmed by the expanded use of Allan variance [1] as an estimate of data scatter.

Thus it is essential to compare various estimates of errors (uncertainties) on the basis of different data models, including statistical and functional representations. In this report a general scheme is outlined, and Allan variance is compared with classic sample variance.

2. General approach

Nowadays in practice the most popular characteristic of data scatter (for the experimental data $x_1, \ldots, x_n$) is the classic sample variance:

$$S^2 = \sum (x_k - \bar{x})^2 /(n-1); \qquad \overline{x} = \sum x_k / n. \qquad (1)$$

It is the best estimate of the variance in the classic case of random sample with constant mean and variance σ^2 under Gaussian distribution, but in other cases one can use some other estimates for the data errors or uncertainties. However, the choice of a proper estimate is not very obvious.

In general, the estimation of error (uncertainty) is a multi-stage procedure [2], which is realised as follows.

1. Selection or determination of the basic model of the error (for instance, random variable, time series, fuzzy set, interval function).
2. Definition of the error characteristic as a scale parameter within the scope of the fixed model.
3. Determination of the estimate for the fixed parameter, which could be applied to the experimental data obtained in measurement.
4. Calculation of the estimate according the chosen formula.

So in the classic case the model of error is a random variable, the parameter (characteristic) is variance, and the estimate is sample variance (1). If one fixes the model of random variable and the variance as parameter, it is possible to use some other estimates of variance, for instance: truncated sample variance $S^2(\alpha)$; median absolute deviation S^2_m; mean absolute deviation S^2_d ; quartile deviation estimate S^2_q [3].

But one can also choose other parameters of random variable distribution, for instance: mean (absolute) deviation; quartile deviation; median deviation. The estimates of the parameters may be taken either those mentioned above, or some more complicate ones.

This simple scheme clearly illustrates the vast set of the parameters, which may be used as error characteristics, and the vast manifold of the estimates for these parameters. Nevertheless, the sample variance is still the most popular one.

But lately another scatter characteristic was proposed - Allan variance [1], evaluated as

$$\sigma_a^2 = \sum (x_{k+1} - x_k)^2 / 2(n-1). \qquad (2)$$

Allan variance proved to be very useful in many measurement problems. Nowadays it is widely used for data processing in measurements, especially in case of flicker or 1/f type noise. It is highly important in the measurements of time and frequency [4].

However, Allan variance is usually introduced as just an empirical value. In this paper two ways are proposed for the formalisation of Allan variance as an important scatter characteristic of data. Allan variance is also compared with classic sample variance for revealing the scope of each estimate.

3. Statistical approach

The first approach is based on the testing of statistical hypotheses. It should be noted that the ratio of Allan variance to the sample variance was used as a test statistic in the classic mean square difference test [5]:

$$\boldsymbol{r = \sigma^2_a / S^2 .}$$

Here null hypothesis H_0 is that $x_1, \ldots, x_n$ is a random sample with constant mean, $Dx_k = \sigma^2$:

$$\boldsymbol{H_0: \; E\, x_k = c.}$$

It could be tested against various alternatives. Usually this test is used in metrology to reveal a systematic shift in data, so the alternative is the following:

$$\boldsymbol{H_1: \; E\, x_{k+1} = E\, x_k + h .}$$

The properties of the ratio r are analysed under hypothesis $\boldsymbol{H_0}$. If $x_1, \ldots, x_n$ is a random sample with Gaussian distribution, there are formulas:

$$\boldsymbol{E\{r\} = 1, \; D\{r\} = (n+2)^{-1} + O(n^{-3}).}$$

Moreover, statistic $(r-1)\,[(n^2-1)/(n-2)]^{1/2}$ is asymptotically standard Gaussian variable.

The critical region is as follows:

$$\boldsymbol{D(\alpha) = \{ r < r_{min}(n, \alpha) \},}$$

and in this case hypothesis $\boldsymbol{H_0}$ is rejected (so systematic errors are present).
The critical value $r_{min}(n, \alpha)$ for $n > 60$ is evaluated by the formula

$$r_{min}(n, \alpha) = 1 + z_\alpha / [\, n + (1 + z_\alpha^2)/2]^{1/2},$$

where z_α - α-quantile of standard Gaussian distribution.

But some other alternative hypothesis may be also considered, first of all - the time series with non-correlated increments:

$$\boldsymbol{H_2: \; x_{k+1} = x_k + z_k .}$$

where $\boldsymbol{z_k}$ - increments with variance $\boldsymbol{Dz_k = \sigma_0^2}$, or innovation series for $\boldsymbol{x_k}$.

The properties of the ratio $\boldsymbol{r}$ under hypotheses stated may be used for comparing the fields of application of two characteristics, Allan variance and sample variance. In particular,

Allan variance σ^2_a is the best estimate of the increment variance σ_0^2 under hypothesis H_2, whereas sample variance S^2 is the best estimate of variance σ^2 under hypothesis H_0.

4. Hilbert space approach

The second approach is based on the vector (functional) representation of time series (or random process).

If R (s, t) is a correlation function on the set T (s, t ∈ T), then H (R) is defined as the reproducing kernel Hilbert space (RKHS) with the kernel R (s, t) [6]. This is a space of the function on the set T with the following properties:

1) all the functions $R_t = R (t, .)$ belong to H (R);

2) scalar product of any f ∈ H (R) to function $R_t = R (t, .)$ is the value of f at t:

$$(f, R_t)_R = f (t) .$$

So the function $R_t = R (t, .)$ defines the linear functional in H (R), giving the value in t.

Random process x (t) with correlation function R (s, t) has a canonical isomorphic representation in the space H (R) with kernel R (s, t): random variable x (t) corresponds to the function $R_t = R (t, .)$, and random variable h ∈ H (x) (belonging to linear span of the random process x (t)) corresponds to the function f ∈ H (R), such as

$$f (t) = (h, x(t))_R = E\, h\, x(t).$$

In particular, for the discrete time series $x_1, \ldots, x_n$ on the finite time interval [1…n] with covariance matrix R (s, t), Hilbert space H (R) is n-dimensional vector space, containing all the vectors of the form $R_t = R (t, .)$, and the scalar product of a vector x ∈ H (R) to $R_t = R (t, .)$ is just t–component of x:

$$(x, R_t)_R = x_t .$$

Functional space H (R) is a convenient tool for investigating of the various properties of random processes. An important advantage of RKHS representations over spectral ones is that RKHS representations are applicable for both stationary and non-stationary processes, and are also valid for generalized random processes (such as white noise or flicker noise). On the other hand, for the stationary processes RKHS representations are directly related with the spectral ones, so all the "spectral" results may be simply formulated in terms of RKHS. This representation is also related with the canonical innovation representations by H.Wald and H.Cramer for non-deterministic time series and random processes.

In particular, if $x_1, \ldots, x_n$ is non-correlated time series (or random sample) with , then RKHS is l^2 -vector space with norm

$$\| x \|^2 = \Sigma \, | x_k |^2.$$

That is numerically coincides with the formula of sample variance (with mean 0).

In the case of generalized white noise process, Hilbert space H (R) is just the space of square-integrable functions L^2 :

$$\text{H (R)} = \{ g: \ \| g \|^2 = \int \, | \, g (t) \, |^2 \, dt < \infty \}.$$

If $x_1, \ldots, x_n$ is time series with non-correlated increments, the corresponding RKHS is the vector space with norm

$$\| \, x \, \|^2 = | \, x_0 \, |^2 + \Sigma \, | \, x_{k+1} - x_k \, |^2$$

That is numerically coincides with the formula of Allan variance (with $x_0 = 0$).

Thus Allan variance σ^2_a is just a norm in RKHS, corresponding to time series with non-correlated increments. So it seems expedient to call it also the increment or innovation variance for time series.

This fact to a certain extent elucidates the mathematical content of Allan variance as the scatter or scale characteristic of time series. In particular, it explains the fact that Allan variance is so useful in the case of signals with non-correlated increments and 1/f type noise.

The mentioned relations of RKHS and spectral representations for the stationary processes provide the opportunities for further investigations of Allan variance properties for various types of noise, including white, flicker noise and some others.

There is also relation of Allan variance with the innovation representations by H.Wald and H.Cramer for non-deterministic time series or random processes, that should be mentioned.

5. Conclusion

Allan variance and sample variance are two distinct scatter or scale characteristic of data, which are generated by different kinds of time series as basic models. In many cases these estimates also correspond to different parameters.

Both characteristics are significant for practice, but they are useful in various cases. Therefore it seems likely that it could sometimes be useful to apply them in common, or choose one of these estimates as preferable one.

It seems helpful to investigate the interrelations of these estimates for wider range of the practically significant models in order to reveal the fields of the combined use and the cases of the preferable use for each estimate.

References

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